Non-Gaussian generalizations of Wick's theorems, related to the Schwinger-Dyson equation.

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Contents

1. Abstract.

In this work we present a number of generalizations of Wick's theorems on integrals with Gaussian weight to a larger class of weights which we call subgaussian. Examples of subgaussian contractions are that of Kac-Moody or Virasoro type, although the concept of a subgaussian weight does not refer a priori to two-dimensional field theory. The generalization was chosen in such a way that the contraction rules become a combinatorical way of solving the Schwinger-Dyson equation. In a still more general setting we prove a relation between solutions of the Schwinger-Dyson equation and a map N, which in the Gaussian case reduces to normal ordering. Furthermore, we give a number of results concerning contractions of composite insertions, which do not suffer from the Johnson-Low problem of "commutation" relations that do not satisfy the Jacobi identity.

2. Introduction.

2.1. Motivation.

2.1.1. *The construction of integration theories.* The motivation of this work comes mainly from the need of defining functional integration. In general one may say that the construction of integration theories proceeds in two steps:

1. First fix the integral of some "elementary" functions. For example the integration theories over $\mathbb{R}^n$ are based on the requirement that
$$Vol([a_1, b_1] \times ... \times [a_n, b_n]) = (b_1 - a_1)..(b_n - a_n).$$
In other words the integral of the characteristic function of $[a_1, b_1] \times .. \times [a_n, b_n]$ is prescribed before having an integration theory.
2. Then, try to extend the notion of integration to more general functions. In the example of $\mathbb{R}^n$, this leads e.g. to the definition of the Lebesgue measure, but also to the original definition of integration as the inverse of differentiation.

Note however that it is not strictly necessary to take characteristic functions as "elementary". For example, one of the typical starting points of functional integration theory over the set of differentiable functions $\phi : \mathbb{R}^n \to \mathbb{R}$ is the following: Let $n \geq 3$ and on the set of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ define the weight $S(\phi) := \int \partial_i \phi \partial^i \phi dx^1..dx^n \in [0, \infty]$. Then whatever the details of the integration theory will be, we require that:
$$\int_{\{\phi:\mathbb{R}^n \to \mathbb{R}\}} e^{-S(\phi)} \phi(x)\phi(y) D\phi = \frac{K}{|x-y|^{n-2}}.$$
The motivation for this starting point is the Schwinger-Dyson equation:

2.1.2. *The Schwinger-Dyson equation.* For fixed $S : \mathbb{R}^D \to \mathbb{R}$, consider the following linear functional:
$$f \mapsto I(f) := \int_{\mathbb{R}^D} f e^{-S} dx^1..dx^D,$$
where f and S are restricted such that it is well defined, and such that upon partial integration boundary terms are zero: In that case the functional satisfies $\forall_{i,f}\ I(\partial_i(S)f) = I(\partial_i f)$, for:
$$0 = \int \frac{\partial}{\partial x^i}(e^{-S} f) dx^1...dx^D = \int e^{-S}(-\partial_i(S)f + \partial_i f) dx^1...dx^D = I(\partial_i f) - I(f \partial_i S).$$
This is the Schwinger-Dyson equation [1] for the functional I. For Gaussian weights, i.e. where S is quadratic, this equation has a unique solution up to normalization for polynomial integrands. Now the point is that even though the equation is motivated by finite dimensional integration, we may also try to solve it in infinite dimensions, and take the solution as a "starting point" for the construction of measures. Appendix B contains a review of the Gaussian Schwinger-Dyson equation, together with the proof of the above given formula for the functional integral.

[1] We will stick to the practice, in the context of functional integration, of calling this formula the Schwinger-Dyson equation, although the above formula was already given for functional integrals by Feynman in [5, formula 45](1948). Another often seen formulation is $\{J_i - (\partial_i S)(\frac{\partial}{\partial J})\} Z(J) = 0$, by setting $Z(J) := I(e^{J_i x^i})$.

Remark 2.1.1. Note that the Schwinger-Dyson equation remains valid if we multiply I by a constant, so if possible we will restrict ouselves to solutions I such that $I(1) = 1$, and such normalized solutions will be denoted by $f \mapsto \langle f \rangle$, in view of the fact that if we set

$$\langle f \rangle := \frac{\int e^{-S} f d\mu}{\int e^{-S} d\mu},$$

then $\langle . \rangle$ is a normalized solution of the Schwinger-Dyson equation. When we say that the solution of the Schwinger-Dyson equation is unique, we will always mean up to normalization.

2.2. **Aim of this work.** We have a number of goals:

1. The first aim of this work is to analyse the conditions for the weight S under which the Schwinger-Dyson equation has a unique solution. For example Gaussian weights have a unique solution, but there is more.
2. Next we aim to extend the notion of normal ordering to non-Gaussian weights in such a way that it is naturally assicated to such weights. The need for such a normal ordering also comes from functional integration: In the Gaussian case, normal ordering can be used to regularise certain types of short distance singularities (see appendix A), and it seems desirable to find the analogue for non-gaussian weights. Furthermore, when using functional integration for geometric purposes, it is essential to only use natural constructions. Therefore one is led to look for naturally constructed normal ordering.
3. We will define the notion of a subgaussian weight for which we will be able to prove a number of generalizations of Wick's theorems [7].
4. Finally we will look for theorems concerning what are called composite insertions. This will be explained in a moment.

2.3. **Overview of the article.**

2.3.1. *Change of variables.* Given a weight $S(x)$, our first step will be to go to a new system of variables $S_i := \partial_{x^i} S$. (Thus, for Gaussian weights $S = \frac{1}{2} g_{ij} x^i x^j$, we just have $S_i = g_{ij} x^j = x_i$. But the only thing we will assume for now is that the S_i's form a coordinate system.) Then the Schwinger-Dyson equation reads:

$$\langle \partial_{i_1}(S) ... \partial_{i_n}(S) \rangle = \sum_{k=2}^{n} \langle \partial_{i_2}(S) .. \partial_{i_1} \partial_{i_k}(S) .. \partial_{i_n}(S) \rangle.$$

Now for Gaussian integrals, the second derivatives $\partial^2 S$ are numbers, so that the above equation becomes a recurrence relation. Our simplest generalization consists in dropping the assumption that $\partial^2 S$ is a number, and replacing it by the assumption that $\partial^2 S$ is at most linear in ∂S. This is what we call the subgaussian case, and it obviously leads to a recurrence relation, so that the subgaussian case has a unique solution too. The next generalization consists in assuming that $\partial^2 S$ can be written as a polynomial in ∂S, and we will call that the polynomial case. For the polynomial case there is also an easy condition which guarantees uniqueness of solutions for the Schwinger-Dyson equation, which is invertibility of normal ordering:

2.3.2. *Normal ordering.* Given a weight S, we define normal ordering [2] inductively using the new variables S_i, as follows: $N(1) := 1$, and

$$N(S_{i_0}..S_{i_n}) := S_{i_0} N(S_{i_1}..S_{i_n}) - \frac{\partial}{\partial x^{i_0}} N(S_{i_1}..S_{i_n}).$$

It is not very difficult to show that if N is invertible, then $\{I$ satisfies the Schwinger-Dyson equation $\Leftrightarrow I(f) = ZN^{-1}(f)I(1)\}$, where Z denotes the projection of polynomial functions of the S_i's on their constant part, e.g. $Z(3 + aS_1 + bS_1S_5) = 3$. This is the main idea of this work.

2.3.3. *The nonabelian case.* In order to make the link with two-dimensional field theory, we will have to generalize the above to the case where instead of using the commuting vectorfields ∂_i, we assume given a not necessarily Abelian Lie algebra L of vectorfields, and instead of the variables $\partial_i S$, we will (roughly) use the variables $T_a S$, where T_a is a basis of that Lie algebra. In section 5 we will prove a generalization of the theorem $\langle . \rangle = ZN^{-1}$ to the non-Abelian case. In this case the definition of normal ordering needs an explicit symmetrization, which will make the proof more complicated:

$$N(X_1(S)...X_n(S)) := \frac{1}{n} \sum_{i=1}^{n} X_i(S) N(X_{[1,n]\setminus i}(S)) - X_i(N(X_{[1,n]\setminus i}(S))).$$

Here $X_{[1,n]\setminus i}(S)$ is shorthand for $X_1(S)..X_n(S)$ with $X_i(S)$ left out.

2.3.4. *Composite insertions and left extensions.* The fact that in $\langle \partial_i(S) s_1..s_n \rangle$ we may eliminate $\partial_i(S)$ in favor of the sum of terms with $\partial_i s_j$ relies on the special form of $\partial_i S$. In general it will not be possible to find a derivation D such that $\langle X(S)Y(S)f \rangle = \langle Df \rangle$. However it <u>may</u> happen sometimes, if we choose X and Y in the right way. In that case we say that with D we have constructed a left extension. (Namely of the corresponding contraction: See section 3 for the definition of contractions. In general an "insertion" is any factor in an expression between brackets $\langle . \rangle$. We will call it composite if it is not a first derivative of S). Section 4 gives examples of such extensions.

2.3.5. *Section by section overview.* In section 3 we will give precise definitions of what we mean by the non-Abelian subgaussian case, and prove a number of theorems concerning them. We will also prove theorems concerning left-extensions in the subgaussian case. Instead of speaking of weights S we will phrase everything in terms of contractions $[. \triangleright .]$, which is a formulation better suited to study the Schwinger-Dyson equation. In that section we will also review a relatively well known algebra as an example of a non-Abelian contraction: The Kac-Moody algebra. In section 4 we will give an example of an infinite dimensional left extension. The example is not new and in fact has a long history, see remark 4.2.3; What we want to emphasize is that the construction is an application of a theorem proved for the general subgaussian case. Finally in section 5 we will be concerned with the proof that $\langle . \rangle = ZN^{-1}$ in the non-Abelian case.

[2] Gaussian normal ordering was introduced in [6]. A number of definitions can be found for other cases besides the Gaussian case, see for example: [16],[20, formula 7b],[21, formula 4], [22, formula 6]. However none of these definitions is directly in terms of the weight S.

2.3.6. *What the article is not about.* One should be aware that even if the solution of the Schwinger-Dyson equation is unique, then this does not replace by any means the notion of integration. Indeed, in the Gaussian case, the solution of the Schwinger-Dyson equation only determines the integral of polynomial integrands. If one is interested in other integrands, then it necessary to go on to what we called step two in the construction of integration theories. Stated precisely:

Definition 2.3.1. *Fix an infinitely differentiable function $S : \mathbb{R}^D \to \mathbb{R}$.*

1. *By the algebra $\mathcal{S}$ of simple functions we then mean the algebra of functions generated by $\{\partial_{i_1}..\partial_{i_n} S | i_j \in \{1, .., D\}, n \geq 1\}$.*
2. *We say that $I : \mathcal{S} \to \mathbb{R}$ is a solution of the Schwinger-Dyson equation iff it is linear and $\forall_{s \in \mathcal{S}}\ I(\partial_i s) = I(s \partial_i S)$.*
3. *I is said to be positive iff $s > 0 \Rightarrow I(s) > 0$.*
4. *A measure $d\mu$ on $\mathbb{R}^D$ is said to be compatible with such a solution iff $s \in \mathcal{S}$ is measurable, and $\int_{\mathbb{R}^D} s d\mu = I(s)$. Such a measure may allow one to integrate other functions besides the $s \in \mathcal{S}$.*

Remark 2.3.2. Note that it is not a good idea to try to extend I "by analyticity" like in $I(x \mapsto \sum_i a_i x^i) := \sum_i a_i I(x^i)$, because this equality need not hold for usual integrals. For example

$$\int e^{-\frac{1}{2}x^2 - x^4} dx \neq \sum_n \frac{1}{n!} \int (-x^4)^n e^{-\frac{1}{2}x^2} dx = \sum_n \frac{(-1)^n (4n-1)!!}{n!} \sqrt{2\pi} = divergent.$$

We will have nothing more to say about measures; This article will restrict its attention to the algebra $\mathcal{S}$. The reader who wishes to know more about the reconstruction and use of functional measures starting from exact Gaussian results is referred to the book by Jaffe and Glimm [23], noting in particular Minlos' theorem (theorem 3.4.2), and the $2D$ non-Gaussian integrals of section 8.6. in that book. Minlos' theorem is also used in Berezin's book [11].

3. Contraction algebras.

Remark 3.0.3. The following definition is motivated as follows: By a contraction we basically mean the expression of $\partial^2 S$ in terms of ∂S. For a Lie algebra of non-Abelian vectorfields with basis $\{T_a\}$, this roughly means that $T_a T_b S$ is expressed in terms of $T_a S$, but not exactly: When using general vectorfields X on some manifold M, we can no more use the fact that the measure $dx^1..dx^n$ on $\mathbb{R}^n$ is invariant under the vectorfields ∂_i that we used before. Therefore there is no point in that case in splitting off the weight e^{-S} from the volume form $\mu = e^{-S}dx^1..dx^n$, so that we will only talk about the volume form μ from now on. By taking Lie derivatives of the integrand, the Schwinger-Dyson equation then reads: $I(X(f)+f\nabla(X)) = 0$, where the divergence is defined by $L_X\mu = \nabla(X)\mu$. When specializing to $\mu = e^{-S}dx^1..dx^n$ and $X = \partial_i$, we see that $\nabla(X) = -X(S)$. So instead of expressing $T_aT_b(S)$ in terms of $T_c(S)$, we will rather be led by the expression of $T_a\nabla(T_b)$ in terms of $\nabla(T_c)$'s.

Recall that the condition that the contraction was polynomial meant that $\partial^2 S$ could be written as a polynomial in ∂S, and that we could decompose elements of $\mathcal{S}$ in a unique way as polynomials in ∂S. This condition is now translated into the fact that the map $X \mapsto X(S)$, or rather $X \mapsto -\nabla(X)$ induces an isomorphism $Sym(L) \to \mathcal{S}$, i.e. the polynomials in the variables S_i in the abelian case now get replaced by elements of $Sym(L)$, the algebra of formal polynomials in elements of the Lie algebra, by the mapping $X \mapsto -\nabla(X)$, $XY \mapsto \nabla(X)\nabla(Y)$, etc. The analogue of the assumption that $\partial^2 S$ can be written as a polynomial in ∂S is that $XY(S)$ is a polynomial in the $X(S)$'s, or more precisely that $X(-\nabla(Y))$ is a polynomial in the $\nabla(X)$'s, i.e. it leads to a map: $[.\triangleright.] : L\otimes L \to Sym(L); [X\triangleright Y] := -X(\nabla(Y))$, which we extend on the right by derivations to $Sym(L)$. By the general property of divergences that $\nabla([X,Y]) = X(\nabla(Y)) - Y(\nabla(X))$, see theorem E.1.4, the contraction map $[.\triangleright.]$ thus obtained will always satisfy the properties

$$[X \triangleright Y] - [Y \triangleright X] = [X,Y] \in Sym(L),$$

$$[X \triangleright [Y \triangleright Z]] - [Y \triangleright [X \triangleright Z]] = [[X,Y] \triangleright Z].$$

This is what motivates the following definition. The subgaussian case, i.e. where $\partial^2 S$ was at most linear in ∂S now corresponds to the contraction being a map $L \otimes L \to K \oplus L \leq Sym(L)$.

Definition 3.0.4. *We define a number of special contraction algebras:*

1. *A polynomial contraction algebra is a Lie algebra with a map* $[.\triangleright.] : L \otimes L \to Sym(L)$*, extended by derivations on the right to* $Sym(L)$ *, which satisfies*
$$[X \triangleright Y] - [Y \triangleright X] = [X,Y] \in L \leq Sym(L),$$
$$[X \triangleright [Y \triangleright Z]] - [Y \triangleright [X \triangleright Z]] = [[X,Y] \triangleright Z],$$
2. *It is called Gaussian if* $[.\triangleright.] : L \otimes L \to K = Sym^0(L)$.
3. *A subgaussian contraction algebra is one in which* $[.\triangleright.] : L \otimes L \to K \oplus L = Sym^{[0,1]}(L)$*. In that case, we extend* $[.\triangleright.]$ *by* $[1 \triangleright 1] := [1 \triangleright X] := 0$*, which makes* $K \oplus L$ *into a pre Lie algebra.* [3]

[3] (I wish to thank C.D.D. Neumann for pointing out the following to me). A pre-Lie algebra is a vectorspace with a bilinear operation $[.\triangleright.]$ satisfying $[a\triangleright[b\triangleright c]] - [b\triangleright[a\triangleright c]] = [[a\triangleright b] - [b\triangleright a]\triangleright c]$, see Gerstenhaber [10, formula 6]. In that case $[a,b] := [a\triangleright b] - [b\triangleright a]$ is a Lie composition.

4. *Normal ordering is the map $N : Sym(L) \to Sym(L)$, defined by $N(1) := 1$, and by*

$$N(X_1...X_n) := \frac{1}{n}\sum_{i=1}^{n} X_i N(X_{[1,n]\setminus i}) - [X_i \triangleright N(X_{[1,n]\setminus i})].$$

5. *The contraction is said to be non-degenerate iff its normal ordering is invertible.*
6. *In that case we set $s_1 * s_2 := N^{-1}(N(s_1)N(s_2))$, which in the Gaussian case corresponds to what is usually called the operator product: $Sym(L) \otimes Sym(L) \to Sym(L)$.*

For a review of Gaussian contractions, see appendix B.

3.1. Subgaussian contraction algebras.

Example 3.1.1. A simple example of a subgaussian contraction is the one-dimensional Abelian Lie algebra with basis element e, and contraction $[e \triangleright e] := \lambda 1 + \mu e$, for some scalars λ and μ.

Theorem 3.1.2. *Set $[X \circ Y] := \frac{1}{2}[X \triangleright Y] + \frac{1}{2}[Y \triangleright X]$. In subgaussian algebras the following holds:*

1.
$$[[X \triangleright Y] \circ Z] + [Y \circ [X \triangleright Z]] - [X \triangleright [Y \circ Z]] = \frac{1}{2}([[Y \triangleright X] \triangleright Z]] + [[Z \triangleright X] \triangleright Y]]).$$

2.
$$N([X \triangleright Y]) = [X \triangleright Y]$$

3.
$$N([X \triangleright Y]Z) = [X \triangleright Y]Z - [[X \triangleright Y] \circ Z].$$

4.
$$[X \triangleright N(YZ)] = N([X \triangleright Y]Z + Y[X \triangleright Z] + \frac{1}{2}[[Y \triangleright X] \triangleright Z] + \frac{1}{2}[[Z \triangleright X] \triangleright Y]).$$

Proof

1. This is an identity which holds in any pre Lie algebra:

$$\begin{aligned} 2.LHS &= [[X \triangleright Y] \triangleright Z] + [Z \triangleright [X \triangleright Y]] + [Y \triangleright [X \triangleright Z]] + [[X \triangleright Z] \triangleright Y] \\ &\quad -[X \triangleright [Y \triangleright Z]] \quad - [X \triangleright [Z \triangleright Y]] \\ &= [[Y \triangleright X] \triangleright Z] + [Z \triangleright [X \triangleright Y]] + [Y \triangleright [X \triangleright Z]] + [[Z \triangleright X] \triangleright Y] \\ &\quad -[Y \triangleright [X \triangleright Z]] - [Z \triangleright [X \triangleright Y]] \\ &= [[Y \triangleright X] \triangleright Z] + [[Z \triangleright X] \triangleright Y]. \end{aligned}$$

2. $[X \triangleright Y] \in Sym^{[0,1]}(L)$.
3. Idem, together with the symmetrized definition of N.
4.

$$\begin{aligned} LHS &= [X \triangleright YZ - [Y \circ Z]] = [X \triangleright Y]Z + Y[X \triangleright Z] - [X \triangleright [Y \circ Z]] \\ &= N([X \triangleright Y]Z) + N(Y[X \triangleright Z]) \\ &\quad +[[X \triangleright Y] \circ Z] + [Y \circ [X \triangleright Z]] - [X \triangleright [Y \circ Z]] = RHS. \end{aligned}$$

□

Remark 3.1.3. I would like to draw particular attention to point 4 above: It is a generalization of the Gaussian fact that $[X \triangleright N(YZ)] = N([X \triangleright YZ])$. In the next section we will make ample use of this formula to handle expressions like $N(\psi(z)\psi(z))$ A more familiar use of the formula is in Gaussian form: It is then the essential statement for proving that in the Gaussian case $\langle N(X_1X_2)N(Y_1Y_2)\rangle$, say, can be expanded in terms of contractions between X's and Y's only; Indeed,

$$\begin{aligned}\langle N(X_1X_2)N(Y_1Y_2)\rangle &= \langle (X_1N(X_2) - [X_1 \triangleright N(X_2)])N(Y_1Y_2)\rangle \\ &= \langle N(X_2)[X_1 \triangleright N(Y_1Y_2)]\rangle = \langle N(X_2)N([X_1 \triangleright Y_1Y_2])\rangle.\end{aligned}$$

Theorem 3.1.4. *(Subgaussian reconstruction.) Let L be finite dimensional Lie algebra. Let $\langle.\rangle : Sym(L) \to K$ be the solution of the Schwinger- Dyson equation determined by a subgaussian contraction on L. Then this contraction can be reconstructed from $\langle.\rangle$ if $c_{ij} := \langle T_iT_j\rangle$ is invertible (T_i a basis for L), as follows: Let $c_{ijk} := \langle T_iT_jT_k\rangle$, and let $[T_i, T_j] =: f_{ij}^k T_k$. Then the contraction is given by $[T_i \triangleright T_j] = g_{ij} + \Gamma_{ij}^k T_k$, where*

$$g_{ij} := c_{ij},$$

$$\Gamma_{ij}^k := \frac{1}{2} g^{kl}(c_{ijl} + f_{ij}^m g_{ml} - f_{il}^m g_{jm} - f_{jl}^m g_{im}).$$

Proof

The proof is similar to the uniqueness proof of the Levi-Cività connection: We know the contraction is subgaussian, so it is of the form $[T_i \triangleright T_j] = g_{ij} + \Gamma_{ij}^k T_k$, and it remains to prove the above relations. Indeed,

$$c_{ij} = \langle T_iT_j\rangle = \langle [T_i \triangleright T_j]\rangle = \langle g_{ij} + \Gamma_{ij}^c T_c\rangle = g_{ij}.$$

Further, we have

$$c_{ijk} = \langle [T_i \triangleright T_j]T_k\rangle + \langle T_j[T_i \triangleright T_k]\rangle = \Gamma_{ij}^m c_{mk} + c_{jm}\Gamma_{ik}^m;$$

Using the fact that $\Gamma_{ij}^k - \Gamma_{ji}^k = f_{ij}^k$, since $\Gamma_{ij}^k T_k - \Gamma_{ji}^k T_k = [T_i \triangleright T_j] - [T_j \triangleright T_i] = [T_i, T_j]$, we arrive at

$$\begin{aligned}c_{ijk} + c_{jik} - c_{kij} &= \Gamma_{ij}^m c_{mk} + \Gamma_{ik}^m c_{jm} + \Gamma_{ji}^m c_{mk} + \Gamma_{jk}^m c_{im} - \Gamma_{ki}^m c_{mj} - \Gamma_{kj}^m c_{im} \\ &= (\Gamma_{ij}^m + \Gamma_{ji}^m)c_{mk} + f_{ik}^m c_{jm} + f_{jk}^m c_{im} \\ &= (2\Gamma_{ij}^m - f_{ij}^m)c_{mk} + f_{ik}^m c_{jm} + f_{jk}^m c_{im}\end{aligned}$$

So that

$$2\Gamma_{ij}^l g_{lk} = c_{ijk} + c_{jik} - c_{kij} + f_{ij}^m c_{mk} - f_{ik}^m c_{jm} - f_{jk}^m c_{im}.$$

□

3.1.1. *Left extensions of a subgaussian contraction.*

Definition 3.1.5. *Let $r \in Sym(L)$. Then we say that a map $[r \triangleright .] : L \to Sym(L)$ is a left extension of the contraction to r, if it satisfies:*

$$\forall_{X\in L}\ [X \triangleright r] - [r \triangleright X] \in L,$$

$$\forall_{X,Y\in L}\ [X \triangleright [r \triangleright Y]] - [r \triangleright [X \triangleright Y]] = [[X \triangleright r] - [r \triangleright X] \triangleright Y].$$

In that case, we make we make $[r \triangleright .]$ act on $Sym(L)$ by derivations. The extension is said to be subgaussian iff $[r \triangleright X] \in K \oplus L$, and $[X \triangleright r] \in K \oplus L$.

Theorem 3.1.6. *If $[r \triangleright .]$ is a subgaussian left extension to r then r behaves in a way similar to elements of L:*

1.
$$\forall_{X\in L, s\in Sym(L)}\ [X\triangleright[r\triangleright s]]-[r\triangleright[X\triangleright s]]=[[X\triangleright r]-[r\triangleright X]\triangleright s].$$
2. *If $\langle.\rangle$ satisfies the Schwinger-Dyson equation then:*
$$\forall_{s\in Sym(L)}\ \langle rs\rangle=\langle r\rangle\langle s\rangle+\langle[r\triangleright s]\rangle.$$
3.
$$[r\triangleright N(YZ)]=N([r\triangleright YZ]+\frac{1}{2}[[Y\triangleright r]\triangleright Z]+\frac{1}{2}[[Z\triangleright r]\triangleright Y]).$$

Proof

1. Since both sides of the defining equation of left extensions concern derivations acting on Y, we can replace Y by s.
2. We prove this by induction on $deg(s)$. If $s=1$ then
$$\langle rs\rangle=\langle r\rangle=\langle r\rangle\langle s\rangle=\langle r\rangle\langle s\rangle+\langle[r\triangleright s]\rangle.$$
Next, assume the identity holds up to $deg(s)$. We will prove it for Xs:
$$\langle rXs\rangle=\langle[X\triangleright rs]\rangle=\langle[X\triangleright r]s\rangle+\langle r[X\triangleright s]\rangle$$
Using that $[X\triangleright r]$ is in $K\oplus L$, and that $[X\triangleright s]$ has degree smaller or equal to that of s:
$$=\langle[[X\triangleright r]\triangleright s]\rangle+\langle[X\triangleright r]\rangle\langle s\rangle+\langle[r\triangleright[X\triangleright s]]\rangle+\langle r\rangle\langle[X\triangleright s]\rangle$$
$$=\langle[[X\triangleright r]\triangleright s]+[r\triangleright[X\triangleright s]]\rangle+\langle[X\triangleright r]\rangle\langle s\rangle+\langle r\rangle\langle Xs\rangle.$$
Thus, using $\langle[X\triangleright r]\rangle=\langle[r\triangleright X]\rangle$, since $[X\triangleright r]-[r\triangleright X]\in L$:
$$\langle rXs\rangle-\langle r\rangle\langle Xs\rangle=\langle[r\triangleright X]\rangle\langle s\rangle+\langle[[X\triangleright r]\triangleright s]+[r\triangleright[X\triangleright s]]\rangle$$
$$=\langle[r\triangleright X]\rangle\langle s\rangle+\langle[[r\triangleright X]\triangleright s]\rangle+\langle[X\triangleright[r\triangleright s]]\rangle$$
$$=\langle[r\triangleright X]s+X[r\triangleright s]\rangle=\langle[r\triangleright Xs]\rangle.$$
3. The proof is identical to that of theorem 3.1.2 by replacing X by r.

□

3.2. **An example from conformal field theory: The Kac-Moody algebra.**

Theorem 3.2.1. *The following almost everywhere defined contraction satisfies the pre-Lie property and therefore defines a subgaussian contraction algebra: The algebra is associated to a Lie algebra L with invariant symmetric bilinear form g, namely it is defined by generating symbols $J(X,z)$, linear in $X\in L$* [4] *, and where $z\in\mathbb{C}$, with contraction:*
$$[J(X,z)\triangleright J(Y,\zeta)]:=\frac{g(X,Y)}{(z-\zeta)^2}1+\frac{J([X,Y],\zeta)}{z-\zeta}.$$

[4] When we say that a symbol is linear in some argument, we mean we impose an equivalence relation on the linear span of these symbols. Thus for example $J(2X,z)\sim 2J(X,z)$, etc.

Proof
The pre-Lie property is equivalent to the statement that $[[a \triangleright b] \triangleright c] + [b \triangleright [a \triangleright c]]$ is symmetric in a and b. This way of proving this property has an advantage over proving that $[[a \triangleright b] \triangleright c] - [a \triangleright [b \triangleright c]]$ is symmetric in a and b, since the former way will automatically give an expression for the three-point function:

$$\langle abc \rangle = \langle [a \triangleright b]c \rangle + \langle b[a \triangleright c] \rangle = \langle [[a \triangleright b] \triangleright c] + [b \triangleright [a \triangleright c]] \rangle.$$

Following this remark, we first prove the following

Lemma 3.2.1.1.

$$[[J(X_1, z_1) \triangleright J(X_2, z_2)] \triangleright J(X_3, z_3)] + [J(X_2, z_2) \triangleright [J(X_1, z_1) \triangleright J(X_3, z_3)]]$$
$$= \frac{g([X_1, X_2], X_3) + J((z_1 - z_3)[[X_1, X_2], X_3] + (z_1 - z_2)[X_2, [X_1, X_3]], z_3)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}.$$

Proof
$LHS =$

$$[\frac{J([X_1, X_2], z_2)}{(z_1 - z_2)} \triangleright J(X_3, z_3)] + [J(X_2, z_2) \triangleright \frac{J([X_1, X_3], z_3)}{(z_1 - z_3)}] = t(g) + t(J),$$

where $t(g)$ denotes the terms involving g, and $t(J)$ those with J.

$$t(g) = \frac{g([X_1, X_2], X_3)}{(z_1 - z_2)(z_2 - z_3)^2}$$
$$+ \frac{g(X_2, [X_1, X_3])}{(z_1 - z_3)(z_2 - z_3)^2} = \frac{g([X_1, X_2], X_3)}{(z_2 - z_3)^2}\left(\frac{1}{z_1 - z_2} - \frac{1}{z_1 - z_3}\right)$$
$$= \frac{g([X_1, X_2], X_3)}{(z_2 - z_3)^2} \frac{z_2 - z_3}{(z_1 - z_2)(z_1 - z_3)} = \frac{g([X_1, X_2], X_3)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)},$$

and

$$t(J) = \frac{J([[X_1, X_2], X_3], z_3)}{(z_1 - z_2)(z_2 - z_3)} + \frac{J([X_2, [X_1, X_3]], z_3)}{(z_1 - z_3)(z_2 - z_3)}$$
$$= \frac{J((z_1 - z_3)[[X_1, X_2], X_3] + (z_1 - z_2)[X_2, [X_1, X_3]], z_3)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)},$$

which proves the lemma.

□

It remains to prove that the result of the lemma is symmetric under the exchange of 1 and 2. This is clear for the term with g, and for the J term, it suffices to prove that

$$(z_1 - z_3)[[X_1, X_2], X_3] + (z_1 - z_2)[X_2, [X_1, X_3]] + (1 \leftrightarrow 2) = 0.$$

Indeed, using the Jacobi identity:

$$LHS = \{(z_1 - z_3) - (z_2 - z_3)\}[[X_1, X_2], X_3]$$
$$+ (z_1 - z_2)\{[X_2, [X_1, X_3]] - [X_1, [X_2, X_3]]\}$$
$$= (z_1 - z_2)[[X_1, X_2], X_3] + (z_1 - z_2)[[X_2, X_1], X_3] = 0.$$

□

Remark 3.2.2. In the same way, the reader may check that the Virasoro algebra also defines a subgaussian contraction; For $c \in \mathbb{R}$, this is an algebra generated by symbols $\partial^k T(z)$, where $k \in \mathbb{N}$ and $z \in \mathbb{C}$. The contraction reads

$$[T(z) \triangleright T(\zeta)] := \frac{c/2}{(z-\zeta)^4} 1 + \frac{2T(\zeta)}{(z-\zeta)^2} + \frac{\partial T(\zeta)}{(z-\zeta)},$$

together with $[\partial^k T(z) \triangleright \partial^l T(w)] := \partial_z^k \partial_w^l [T(z) \triangleright T(w)]$.

Definition 3.2.3. *A module for a pre-Lie algebra is defined to be a module for the induced Lie-algebra. The reader may check that the following operations define modules for the Virasoro and Kac-Moody algebras:*

$$[T(z) \triangleright \phi(\zeta)] := \frac{h.\phi(\zeta)}{(z-\zeta)^2} + \frac{\partial \phi(\zeta)}{(z-\zeta)}.$$

$$[J(X,z) \triangleright \phi(v,\zeta)] := \frac{J(Xv,\zeta)}{z-\zeta}.$$

Here, h is some fixed number, $\zeta \in \mathbb{C}$ and v runs linearly over a representation space of the Lie algebra L. In that case the symbol ϕ is called a primary field for T or J, and the number h is called its weight.

4. Examples of left extensions and contraction morphisms.

4.1. **The construction of left extensions.** It is in general difficult to find left exensions $[r \triangleright .]$ for a contraction.

Here we will describe a method to guess $[r \triangleright X]$ from $[X \triangleright r]$ in special cases. The cases we are thinking about are those where the Lie algebra has a basis of symbols $\phi_i(x)$ and their derivatives, where x runs over $\mathbb{R}^D$, and a contraction of the form

$$[\phi_i(x) \triangleright \phi_j(y)] = c_{ij}^k(x-y)\phi_k(y).$$

(Or possibly also involving derivatives in the right hand side). From the fact that $\langle \phi_i(x)\phi_j(y)..\rangle = \langle \phi_j(y)\phi_i(x)..\rangle$ one concludes that $[\phi_i(x) \triangleright \phi_j(y)] - [\phi_j(y) \triangleright \phi_i(x)]$ is regular in $(x-y)$, since the only singularities in $(x-y)$ in the expectation value $\langle \phi_i(x)\phi_j(y)..\rangle$ come from $[\phi_i(x) \triangleright \phi_j(y)]$. If on top of that we know that $c_{ij}^k(x-y)$'s are singular algebraic functions of $(x-y)$, then using symbolic Taylor expansions like $\phi(x+h) = \phi(x) + h^i(\partial_i\phi)(x) + ..$, one can determine $[\phi_i(x) \triangleright \phi_j(y)]$ from $[\phi_j(y) \triangleright \phi_i(x)]$: Indeed, modulo regular terms, we then have

$$[\phi_i(x) \triangleright \phi_j(y)] = [\phi_j(y) \triangleright \phi_i(x)] = c_{ji}^k(y-x)\phi_k(x)$$

$$= c_{ji}^k(y-x)\sum_{n=0}^{\infty}\frac{1}{n!}[(x^i-y^i)\partial_i]^n\phi_k(y).$$

Thus from this we can read off $[\phi_i(x) \triangleright \phi_j(y)]$ modulo regular terms. But since we know that this contraction has only singular coefficients, the contraction is determined. We can use the same rules to produce a candidate for $[r \triangleright X]$ from $[X \triangleright r]$ if r is some higher order element in $Sym(L)$. There is however no guarantee that this will actually give a left extension in the sense of definition 3.1.5. We will give examples from 2D holomorphic field theory, where the procedure actually gives an extension. It leads e.g. to an embedding of the Kac-Moody algebra in the $Sym(L)$ of free fermion fields.

Remark 4.1.1. A number of remarks seem to be in place to avoid loss of time for the reader:

1. It seems that the above procedure does not have an analogue in an arbitrary subgaussian contraction algebra, since even if we start out with a contraction which does not involve the sign ∂, then still the left-extended contraction may involve that sign. (See the Sugawara construction below.)
2. Unfortunately, the procedure that we use to produce left extensions in certain 2D contraction algebras, does not seem useful in higher dimension: Although we can still apply the same method, I haven't found an example that satisfies the properties of 3.1.5. This is very reminiscent of the work by Johnson and Low, who in the operator language found that anomalous current "commutation relations" do not satisfy the Jacobi identity in higher dimensons. Here one can see the advantage of the contraction language over the operator language, since there is no inconsistency in the sole fact of not being able to find left extensions of a contraction. (It does however remain a <u>challenge</u> to actually find one. Also it would be interesting to see any subgaussian contraction in higher dimensions, apart from the Gaussian ones.)

4.2. **The Jordan and Sugawara constructions.** We will here repeat some relatively old constructions, using on the one hand the two-dimensionality which provides a way to construct the left extensions, and some general formulae for subgaussian contraction algebras. There is nothing original about the constructions themselves: We only wish to comunicate the use of subgaussian formulae.

4.2.1. *Jordan-Tomonaga-Coleman-Gross-Jackiw: $J \mapsto N(\psi\psi)$ in 2D.*

Theorem 4.2.1. *Let (V, g) be a finite-dimensional orthogonal representation space for a Lie algebra L. Set $g_V(X, Y) := Tr_V(XY)/2$. Let F_V be the contraction algebra on odd generators $\psi(v, z)$ and Gaussian contraction $[\psi(v, z) \triangleright \psi(w, \zeta)] := \frac{g(v,w)}{z-\zeta}$. Then we have a map $Kac - Moody(L, g_V) \to F_V$, as follows* [5]*:*

$$J(X, z) \mapsto \frac{1}{2} X_{\alpha\beta} N(\psi(e^\alpha, z)\psi(e^\beta, z)).$$

It satisfies:

1. $$[\psi(v, z) \triangleright J(X, \zeta)] = -\frac{\psi(Xv, \zeta)}{z - \zeta}.$$
2. $$[J(X, z) \triangleright \psi(v, \zeta)] = \frac{\psi(Xv, \zeta)}{z - \zeta}.$$
3. $$[J(X, z) \triangleright J(Y, \zeta)] = \frac{Tr(XY)}{2(z - \zeta)^2} + \frac{J([X, Y], \zeta)}{z - \zeta}.$$

Proof

1. We use the notation $Xe_\alpha = X^\beta{}_\alpha e_\beta$. Then $X_{\alpha\beta} = -X_{\beta\alpha}$ by orthogonality. Further, since ψ is odd, we get a number of extra minus signs that we did not include in our previous discussion. See remark F.0.13. Using the formulas for Gaussian contractions:

$$LHS = \frac{1}{2} X_{\alpha\beta} N([\psi(v, z) \triangleright \psi(e^\alpha, \zeta)]\psi(e^\beta, \zeta))$$
$$-\frac{1}{2} X_{\alpha\beta} N(\psi(e^\alpha, \zeta)[\psi(v, z) \triangleright \psi(e^\beta, \zeta)])$$
$$= \frac{1}{2} X_{\alpha\beta}\{\frac{v^\alpha}{z - \zeta}\psi(e^\beta, \zeta) - \psi(e^\alpha, \zeta)\frac{v^\beta}{z - \zeta}\}$$
$$= \frac{1}{2(z - \zeta)}\{-\psi(v^\alpha X_{\beta\alpha} e^\beta, \zeta) - \psi(v^\beta X_{\alpha\beta} e^\alpha, \zeta)\}$$
$$= \frac{-1}{z - \zeta}\psi(v^\alpha X^\beta{}_\alpha e_\beta, \zeta) = \frac{-1}{z - \zeta}\psi(v^\alpha X(e_\alpha), \zeta) = \frac{-\psi(Xv, \zeta)}{z - \zeta}.$$

[5] This needs some explanation since $[\psi(z) \triangleright \psi(z)]$ is undefined. We will not use any specific value for this contraction however: We will only use formula 4 of theorem 3.1.2. This is really a theorem about pre Lie algebras P if we define normal ordering to be $Sym(P) \to Sym(P)$. Thus, to define the above calculus with undefined contractions, we go to the universal pre Lie algebra on symbols $\psi^\alpha(z), 1$, and impose the relation $[\psi(z) \triangleright \psi(w)] = 1/(z - w)$ only for $z \neq w$. Thus, $[\psi(z) \triangleright \psi(z)]$ remains a symbol.

2. This is where we construct a left extension, using the rules that we explained before, i.e. we do the calculation modulo regular terms, and require commutators to be regular.
$$LHS = [\psi(v,\zeta) \triangleright J(X,z)] = \frac{-\psi(Xv,z)}{\zeta - z} = \frac{-\psi(Xv,\zeta)}{\zeta - z} = RHS.$$
We ask the reader to check for himself that this really gives a left extension in the sense we defined it.
3. Here we will use (the super version of) theorem 3.1.2:
$$LHS = [J(X,z) \triangleright N(\frac{1}{2}Y_{\alpha\beta}\psi^\alpha(\zeta)\psi^\beta(\zeta))]$$
$$= N(\frac{1}{2}Y_{\alpha\beta}[J(X,z) \triangleright \psi^\alpha(\zeta)]\psi^\beta(\zeta)) + N(\frac{1}{2}Y_{\alpha\beta}\psi^\alpha(\zeta)[J(X,z) \triangleright \psi^\beta(\zeta)])$$
$$-N(\frac{1}{4}Y_{\alpha\beta}[[\psi^\beta(\zeta) \triangleright J(X,z)] \triangleright \psi^\alpha(\zeta)]) + N(\frac{1}{4}Y_{\alpha\beta}[[\psi^\alpha(\zeta) \triangleright J(X,z)] \triangleright \psi^\beta(\zeta)])$$
$$= N(\frac{1}{2}Y_{\alpha\beta}\frac{\psi(Xe^\alpha,\zeta)}{z-\zeta}\psi^\beta(\zeta)) + N(\frac{1}{2}Y_{\alpha\beta}\psi^\alpha(\zeta)\frac{\psi(Xe^\beta,\zeta)}{z-\zeta})$$
$$-N(\frac{1}{4}Y_{\alpha\beta}[-\frac{\psi(Xe^\beta,z)}{\zeta - z} \triangleright \psi(e^\alpha,\zeta)]) + N(\frac{1}{4}Y_{\alpha\beta}[-\frac{\psi(Xe^\alpha,z)}{\zeta - z} \triangleright \psi(e^\beta,\zeta)])$$
$$= \frac{Y_{\alpha\beta}X_\gamma{}^\alpha}{2(z-\zeta)}N(\psi^\gamma(\zeta)\psi^\beta(\zeta)) + \frac{Y_{\alpha\beta}X_\gamma{}^\beta}{2(z-\zeta)}N(\psi^\alpha(\zeta)\psi^\gamma(\zeta))$$
$$+\frac{Y_{\alpha\beta}}{4(\zeta - z)}\frac{(Xe^\beta)^\alpha}{z-\zeta} - \frac{Y_{\alpha\beta}}{4(\zeta - z)}\frac{(Xe^\alpha)^\beta}{z-\zeta}$$
Now $Y_{\alpha\beta}X_\gamma{}^\alpha = (XY)_{\gamma\beta}$, and $Y_{\alpha\beta}X_\gamma{}^\beta = Y_\alpha{}^\beta X_{\gamma\beta} = -Y_\alpha{}^\beta X_{\beta\gamma} = -(YX)_{\alpha\gamma}$, so we get
$$= \frac{(XY)_{\gamma\beta}}{2(z-\zeta)}N(\psi^\gamma(\zeta)\psi^\beta(\zeta)) - \frac{(YX)_{\alpha\gamma}}{2(z-\zeta)}N(\psi^\alpha(\zeta)\psi^\gamma(\zeta))$$
$$-\frac{Y_{\alpha\beta}}{4(z-\zeta)^2}(X^{\alpha\beta} - X^{\beta\alpha}) = RHS.$$

□

4.2.2. *Sugawara-Coleman-Gross-Jackiw: Nonabelian case of $T \mapsto N(JJ)$.*

Theorem 4.2.2. *Let (L,g) be a finite dimensional reductive Lie algebra, with invariant metric, such that $ad(T_aT^a) = 2c^\vee id_L$. Consider the following map from the Virasoro algebra to the Kac-Moody algebra of (L,kg):*
$$T(z) := \frac{1}{2k + 2c^\vee}N(J(T^a,z)J(T_a,z)).$$
It satisfies:

1. $$[J(X,z) \triangleright T(\zeta)] = \frac{J(X,\zeta)}{(z-\zeta)^2}.$$
2. $$[T(z) \triangleright J(X,\zeta)] = \frac{J(X,\zeta)}{(z-\zeta)^2} + \frac{\partial J(X,\zeta)}{z-\zeta}.$$
3. $$[T(z) \triangleright T(\zeta)] = \frac{k|L|}{k+c^\vee}\frac{1/2}{(z-\zeta)^4} + \frac{2T(\zeta)}{(z-\zeta)^2} + \frac{\partial T(\zeta)}{(z-\zeta)}.$$

Proof
Making use of $ad(T_aT^a) = 2c^\vee id_L$ and

$$[T_a, T^b]^c T_b = g(T^c, [T_a, T^b])T_b = g([T^c, T_a], T^b)T_b = [T^c, T_a],$$

gives:

1.

$$(2k + 2c^\vee)[J_a(z) \triangleright T(\zeta)] = [J_a(z) \triangleright N(J^b(\zeta)J_b(\zeta))] = N([J_a(z) \triangleright J^b(\zeta)]J_b(\zeta)$$
$$+J^b(\zeta)[J_a(z) \triangleright J_b(\zeta)]) + \frac{1}{2}N([[J^b(\zeta) \triangleright J_a(z)] \triangleright J_b(\zeta)] + [[J_b(\zeta) \triangleright J_a(z)] \triangleright J^b(\zeta)])$$
$$= N(2\{\frac{kg_{ab}}{(z-\zeta)^2} + \frac{J([T_a, T_b], \zeta)}{z-\zeta}\}J^b(\zeta) + [\frac{J([T_b, T_a], z)}{\zeta - z} \triangleright J^b(\zeta)])$$

$= \frac{2kJ_a(\zeta)}{(z-\zeta)^2} + A + B$, where

$$(z-\zeta)A = 2N(J([T_a, T_b], \zeta)J^b(\zeta)) = 2N(J(T_c, \zeta)J([T_a, T_b]^c T^b, \zeta))$$
$$= 2N(J_c(\zeta)J([T^c, T_a], \zeta)) = -A(z-\zeta) = 0,$$

and

$$B = \frac{1}{z-\zeta}\{\frac{g([T_a, T_b], T^b)}{(z-\zeta)^2} + \frac{J([[T_a, T_b], T^b], \zeta)}{z-\zeta}\} = 0 + \frac{2c^\vee J(T_a, \zeta)}{(z-\zeta)^2}.$$

This gives the right hand side.

2. Calculus modulo regular terms, with $X = T_a$:

$$LHS = [J_a(\zeta) \triangleright T(z)] = \frac{J_a(z)}{(z-\zeta)^2} = \frac{J_a(\zeta) + (z-\zeta)\partial J_a(\zeta)}{(z-\zeta)^2} = RHS.$$

3.

$$[T(z) \triangleright N(J_a(\zeta)J^a(\zeta))] = N(2[T(z) \triangleright J_a(\zeta)]J^a(\zeta) + [[J_a(\zeta) \triangleright T(z)] \triangleright J^a(\zeta)])$$
$$= N(\frac{2J_a(\zeta)J^a(\zeta)}{(z-\zeta)^2} + \frac{2\partial J_a(\zeta)J^a(\zeta)}{z-\zeta} + [\frac{J_a(z)}{(\zeta - z)^2} \triangleright J^a(\zeta)])$$
$$= \frac{2}{(z-\zeta)^2}N(J_a(\zeta)J^a(\zeta)) + \frac{1}{z-\zeta}\partial N(J_a(\zeta)J^a(\zeta))$$
$$+\frac{1}{(\zeta - z)^2}\{\frac{kg_{ab}g^{ab}}{(z-\zeta)^2} + \frac{J([T^a, T_a], \zeta)}{z-\zeta}\}$$
$$= (2k + 2c^\vee)\{\frac{2T(\zeta)}{(z-\zeta)^2} + \frac{\partial T(\zeta)}{z-\zeta}\} + \frac{k|L|}{(z-\zeta)^4}.$$

□

Historical remark 4.2.3. This type of extensions has a number of original motivations: First, in four-dimensional field theory of the early 1930's scientists following de Broglie [1, formula 2] were trying to see if the photon field could be described as bilinear in the field of the newly discovered neutrino. And second, scientists tried to describe many-electron systems in terms of a formal "soundwave" bilinear in the electron field [2, formula 16]. P.Jordan gave a number of derivations that bilinear combinations of the neutrino field in two dimensions give operators which satisfy canonical bosonic commutation relations [3, formulas 6,15]. However at that time the normal ordering procedure had not been invented, and Jordan's claim that the bilinear construction was possible in two dimensions was not universally accepted, see e.g. the articles by V. Fock [4] on the subject, who advocated the a priori impossiblilty of the construction. Gaussian normal ordering was introduced in 1949,

in [6]. In 1950, Tomonaga [8] gave a new derivation of the bilinear construction without normal ordering, but trying to account for a number of steps by considering them as good approximations. New problems with bilinear constructions were found by Johnson and Low [12, section 6] in 1966, namely that in higher dimensions, the equal-time "commutation relations" of composite operators do not satisfy the Jacobi identity. (Which by the way means that they can't be commutation relations of operators. We will see that in the context of contraction algebras, something similar happens which however is not logically fatal like the problems with the operator language.) In the 1960's it became fashionable try to describe elementary particles only by the currents J of their conserved quantities: Sugawara [14, formula 8] suggested in 1968 to generalize the Euler relation $T_{\mu\nu} := J_\mu J_\nu - \frac{1}{2} g_{\mu\nu} J_\rho J^\rho$ to the non-Abelian and quantum mechanical case, in four dimensions, and without including normal ordering or any regularization in his discussion. Coleman, Gross and Jackiw [15] subsequently discovered some paradoxes, but showed that they disappear when the construction is regularized. They also discovered that Sugawara's construction was better behaved in two dimensions than in four. In 1984, an analysis of the Sugawara construction for any reductive Lie algebra was given by Knizhnik and Zamolodchikov [19].

Here are some more contraction morphisms which the reader may check as an exercise:

1. Starting from the free fermions of theorem 4.2.1, set
$$T(z) := \frac{1}{2} g_{\alpha\beta} N(\partial\psi^\alpha(z)\psi^\beta(z)).$$
This defines a morphism from Virasoro with $c = |V|/2$.
2. Starting with the semidirect product of Kac-Moody with Virasoro and for $Q \in L$, set
$$\tilde{T}(z) := T(z) + \partial J(Q, z).$$
Find conditions under which $\tilde{T}$ is itself of Virasoro type.
3. Starting with $[b(z) \triangleright c(w)] := -[c(w) \triangleright b(z)] := 1/(z-w)$, and $[b \triangleright b] := [c \triangleright c] := 0$, set
$$T(z) := N\{p(\partial b)(z)c(z) + qb(z)\partial c(z)\}.$$
Check that this gives a Virasoro algebra iff $p - q = 1$, that in that case we have $c = -2(p^2 + q^2 + 4pq)$, $h_b = -q$ and $h_c = p$, where h is the number occuring in the definition 3.2.3.

4.3. **Remarks on higher dimensional left extensions.**

Definition 4.3.1. *We define a Gaussian contraction on odd symbols $\{\psi_{A\alpha}(x)\}$, where $x \in \mathbb{R}^D$, $\{e_A\}$ and $\{e_\alpha\}$ are bases of vectorspaces, as follows*[6]*:*
$$[\psi_{A\alpha}(x) \triangleright \psi_{B\beta}(y)] := (\gamma_\mu)_{AB} k_{\alpha\beta} f^\mu(x, y) := (\gamma_\mu)_{AB} k_{\alpha\beta} \frac{x^\mu - y^\mu}{|x - y|^2}.$$
Next, for Γ in the Clifford algebra of $\mathbb{R}^D$, and matrix X, we set:
$$J(\Gamma \otimes X, x) := \frac{1}{2} X^{\alpha\beta} \Gamma^{AB} N(\psi_{A\alpha}(x)\psi_{B\beta}(x)).$$

[6]Beware that for $D \neq 2$ this is not the contraction induced by the Dirac action in D dimensions since that one is proportional to $(\gamma_\mu)_{AB} \frac{x^\mu - y^\mu}{|x-y|^D}$. We use this contraction only as an illustration because it is easier to handle.

Theorem 4.3.2. *J satisfies the following properties, where A^T denotes the transpose of A.*

1.
$$J(\Gamma^T \otimes X^T, x) = -J(\Gamma \otimes X, x).$$

2.
$$[\psi(s \otimes v, y) \triangleright J(\Gamma \otimes X, s)] = \frac{1}{2} f_\mu(y, x)(\psi(\Gamma^T \gamma^\mu s \otimes X^T v, x) - \psi(\Gamma \gamma^\mu s \otimes X v, x)).$$

3.
$$[J(\Gamma \otimes X, x) \triangleright \psi(s \otimes v, y)] = \frac{1}{2} f_\mu(x, y)(\psi(\Gamma \gamma^\mu s \otimes X v, y) - \psi(\Gamma^T \gamma^\mu s \otimes X^T v, y)).$$

4.
$$[J(\Gamma \otimes X, x) \triangleright J(\Delta \otimes Y, y)] = \langle J(\Gamma \otimes X, x) J(\Delta \otimes Y, y) \rangle + \frac{1}{2} f_\mu(x, y) \times$$
$$J(\Gamma \gamma^\mu \Delta \otimes XY - \Gamma^T \gamma^\mu \Delta \otimes X^T Y - \Gamma \gamma^\mu \Delta^T \otimes X Y^T + \Gamma^T \gamma^\mu \Delta^T \otimes X^T Y^T, y).$$

Remark 4.3.3. The above is is an illustration of what a subgaussian contraction in higher dimensions might have looked like if it really satisfied the pre Lie property, but it doesn't. Note that J here is not really the Noether current corresponding to the given contraction, but it is my experience that even if one restricts to Noether currents, the Pre-Lie property is not satisfied.

5. An existence theorem.

In this section we will be concerned with the solution of the Schwinger-Dyson equation in the polynomial nonabelian case. Our first aim will be to prove that just like in the abelian case, the solution is necessarily given by ZN^{-1}. I.e. we start with uniqueness, which is easy:

Theorem 5.0.4. *(Uniqueness). Let I satisfy the Schwinger-Dyson equation. Then $\forall_{s\in Sym(L)} IN(s) = Z(s)I(1)$. So if N is invertible, then $I = I(1).ZN^{-1}$.*

Proof
For $deg(s) = 0$, this equation reads $sI(1) = sI(1)$, and for higher degree, we have:

$$IN(X_1..X_n) = \frac{1}{n}\sum_{i=1}^{n} I(X_i N(X_{[1,n]\backslash i}) - [X_i \triangleright N(X_{[1,n]\backslash i})]) = 0 = Z(X_1..X_n).$$

□

Remark 5.0.5. Thus, to prove existence it suffices to prove that ZN^{-1} satisfies the Schwinger-Dyson equation, i.e. we have to prove that $ZN^{-1}(Xs - [X\triangleright s]) = 0$. Let us start with the easiest case in order to see what exactly the difficulties are: The case $s = Y \in L$. Then we have to prove that $ZN^{-1}(XY - [X \triangleright Y]) = 0$. Now

$$XY - [X \triangleright Y] = N(XY) + \frac{1}{2}[X \triangleright Y] + \frac{1}{2}[Y \triangleright X] - [X \triangleright Y] = N(XY + \frac{1}{2}[Y, X]),$$

So that indeed $ZN^{-1}(XY - [X \triangleright Y]) = 0$. Now we have to try to generalize this procedure to arbirary s. Our first step will be to prove the formula for $N(s)$ instead of s, which is the same if N is invertible: I.e. we will prove $ZN^{-1}(XN(s) - [X \triangleright N(s)]) = 0$. This version is better suited for proof by induction, in view of the definition of normal ordering. So the question is: Given X and s, can we construct $R(X, s)$ such that $XN(s) - [X \triangleright N(s)] = N(R)$ and $Z(R) = 0$? Now in the Abelian case this was easy for we could take $R(X, s) := Xs$, however we are in a more complicated situation now because of the symmetrization in the definition of N: What we saw above is that $R(X, Y) = XY + [Y, X]/2$. It turns out that it is possible to find such an R for all s in the nonabelian case, which is the main result of theorem 5.2.3. We will also be proving a number of extra identities that are useful for a slightly generalized case of the Schwinger-Dyson equation, namely where boundary terms are not assumed to be zero, but instead are assumed to be given by some presecribed (possibly nonzero) map J.

Remark 5.0.6. We will be doing calculations in $Sym(L)$ throughout, but note that the calculations involving contractions apply in a weakened form to the nonpolynomial contractions of appendix D if we allow N and ν to take values in the universal contraction algebra $UEC(L)$ defined in that appendix.

5.1. **Preliminaries on the symmetric algebra of a Lie algebra L.**

Remark 5.1.1. We will be defining maps on $Sym(L)$ without going through the explicit symmetrization every time. To that end we include the following theorem. It is, say, the statement that in symmetric algebras every element of $Sym^n(V)$ can be written as a sum of n-th powers, for example $2XY = (X + Y)^2 - X^2 - Y^2$. This will simplify matters when proving that $ZN^{-1}(XN(s) - [X \triangleright N(s)]) = 0$, because we will only prove that $ZN^{-1}(XN(Y^n) - [X \triangleright N(Y^n)]) = 0$, which as we will see is easier.

Theorem 5.1.2. *(Polarization.) Let V, W be vectorspaces, $G : V^{\otimes n} \to W$ linear, then there is a unique linear map $G_s : Sym^n(V) \to W$ such that $\forall_{v \in V} G_s(v^n) = G(v^n)$.*

Proof
Existence is evident from the following example:

$$G_s(X_1..X_n) := \frac{1}{n!} \sum_{\sigma \in S_n} G(X_{\sigma(1)}, .., X_{\sigma(n)}).$$

Next, we have the following formula in $Sym(V)$:

$$n! X_1 ... X_n = \sum_{S \subset \{1,...,n\}} (-1)^{n-|S|} (\sum_{s \in S} X_s)^n,$$

which is proved by noting that both sides are symmetric and homogeneous polynomials which can be divided by X_1, so that both sides are equal up to scalar multiplication. To determine this factor, we take $X_1 = X_2 = .. = X_n$, and use that $\sum_{k=0}^n (-1)^{n-k} k^n \binom{n}{r} = n!$ This gives uniqueness, since

$$G_s(X_1..X_n) = \frac{1}{n!} \sum_{S \subset \{1,...,n\}} (-1)^{n-|S|} G_s(\sum_{s \in S} X_s)^n$$
$$= \frac{1}{n!} \sum_{S \subset \{1,...,n\}} (-1)^{n-|S|} G(\sum_{s \in S} X_s)^n$$

.

□

Definition 5.1.3. *Let L be a Lie algebra. Define the following maps:*

1. $Z : Sym(L) \to K \leq Sym(L)$, *"Zero degree projection".*
2. $M : L \otimes Sym(L) \to Sym(L)$; *"Multiply":*
$$M(X \otimes s) := Xs.$$
3. $S : Sym(L) \to L \otimes Sym(L)$; *"Split":*
$$S(1) := 0;\ S(X_1...X_n) := \frac{1}{n} \sum_{i=1}^n X_i \otimes X_{[1,n] \setminus i}.$$
4. $\Sigma : L \otimes Sym(L) \to L \otimes Sym(L)$; *"Symmetrize":*
$$\Sigma(X_0 \otimes X_1..X_n) := \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} X_{\sigma(0)} \otimes X_{\sigma(1)}..X_{\sigma(n)}.$$
5. $r := \oplus_n r_n : L \otimes Sym^n(L) \to Sym^{[1,n]}(L)$; *"Rest term", and* $C := \oplus_n C_n : L \otimes Sym^n(L) \to L \otimes Sym^{[0,n-1]}(L)$ *"Commutator term", inductively as follows:* [7]

$$r(X, 1) := 0;\ C(X, 1) := 0,$$
$$C(X, Y^{n+1}) := \frac{n+1}{n+2} \{[Y, X] \otimes Y^n + Y \otimes r(X, Y^n)\}.$$
$$r(X, Y^{n+1}) := \frac{n+1}{n+2} \{[Y, X] Y^n + Y r(X, Y^n) + r([Y, X], Y^n) + r(Y, r(X, Y^n))\}.$$

[7]The motivation for the definition of r comes from the proof theorem 5.2.3: It is chosen in such a way the equality referred to in footnote 9 holds. The definiton of C is useful since we then have $r = (M + r)C$, as proved in the next theorem.

6. $M_\lambda := M + \lambda r$;*"Modified multiplication"*.
7. $\pi_\lambda := SM_\lambda$, *"Projection"*. [8]

Theorem 5.1.4. *These maps satisfy the following properties:*

1. $M_\lambda = M + \lambda r$, $\pi_\lambda = SM_\lambda$.
2. $r = (M + r)C$.
3. $\Sigma^2 = \Sigma$.
4. $\Sigma = SM = \pi_0$.
5. $Zr = ZM = 0$.
6. $CS = 0$, $rS = 0$.
7. $\pi_\lambda \Sigma = \Sigma$.
8. $\pi_\lambda = \lambda\Sigma + \pi_\lambda C$.
9. $\pi_\lambda^2 = \pi_\lambda$.

Proof

1. By definition.
2.

$$r(X, Y^{n+1}) = \frac{n+1}{n+2}\{[Y, X]Y^n + Yr(X, Y^n) + r([Y, X], Y^n) + r(Y, r(X, Y^n))\}$$

$$= \frac{n+1}{n+2}(M + r)\{[Y, X] \otimes Y^n + Y \otimes r(X, Y^n)\} = (M + r)C(X, Y^{n+1}).$$

3. Left to the reader.
4. By polarization it suffices to prove that $\Sigma(X \otimes Y^n) = SM(X \otimes Y^n)$. Indeed:

$$LHS = \frac{1}{(n+1)}(X \otimes Y^n + nY \otimes XY^{n-1}) = S(XY^n) = RHS.$$

5. $ZM(X \otimes s) = Z(Xs) = 0$. Next, we prove by induction on $|s|$ that $Zr(X, s) = 0$. Indeed, $Zr(X, 1) = Z(0) = 0$; Suppose that the identity holds up to degree n. Then we have:

$$Zr(X, Y^{n+1}) = Z(M + r)C(X, Y^{n+1}) = (Zr)C(X, Y^{n+1}) = 0,$$

by induction since C lowers degree.

6. We will prove by induction on $|s|$ that $CS(s) = 0$ and $rS(s) = 0$. Indeed $CS(1) = C(0) = 0$, $rS(1) = r(0) = 0$, $CS(X) = C(X \otimes 1) = 0$, and $rS(X) = r(X \otimes 1) = 0$. So assume these identities hold up to degree $n + 1$. Then:

$$CS(X^{n+2}) = C(X \otimes X^{n+1}) = \frac{n+1}{n+2}X \otimes r(X, X^n) = \frac{n+1}{n+2}X \otimes rS(X^{n+1}) = 0,$$

and

$$rS(X^{n+2}) = (M + r)CS(X^{n+2}) = 0.$$

7. $\pi_\lambda \Sigma = S(M + \lambda r)\Sigma = \Sigma^2 + \lambda SrSM = \Sigma^2 = \Sigma$.
8. $\pi_\lambda = S(M + \lambda r) = SM + \lambda S(M + r)C = \lambda SM + S(M + \lambda r)C = \lambda\Sigma + \pi_\lambda C$.

[8] This map with $\lambda = 1$ is used in theorem 5.3.4. In that theorem a condition of the form $J = J\pi$ appears, which motivates us to try to prove that $\pi^2 = \pi$. This is indeed the case, as is demonstrated in the next theorem.

9. We will prove by induction on $|s|$ that $\pi_\lambda^2(X\otimes s) = \pi_\lambda(X\otimes s)$. Indeed, $\pi_\lambda(X\otimes 1) = X\otimes 1 = \pi_\lambda^2(X\otimes 1)$. Next, assume that the identity holds up to degree n, and that $|s| = n+1$. Then since C lowers degree, we have $\pi_\lambda^2 C(X\otimes s) = \pi_\lambda C(X\otimes s)$, so that:
$$\begin{aligned}\pi_\lambda^2(X\otimes s) &= \pi_\lambda(\lambda\Sigma+\pi_\lambda C)(X\otimes s) = \lambda\pi_\lambda\Sigma(X\otimes s)+\pi_\lambda^2 C(X\otimes s)\\ &= \lambda\Sigma(X\otimes s)+\pi_\lambda C(X\otimes s) = (\lambda\Sigma+\pi_\lambda C)(X\otimes s) = \pi_\lambda(X\otimes s).\end{aligned}$$
$\square$

5.2. Theorems involving contractions.

Definition 5.2.1. *Given a polynomial contraction on L, we define the following maps: $N : Sym(L) \to Sym(L)$, "Normal ordering", and $\nu : L\otimes Sym(L) \to Sym(L)$, "Greek normal ordering", inductively as follows:*
$$N(1) := 1;\ N(X_1...X_n) := \frac{1}{n}\sum_{i=1}^{n} X_i N(X_{[1,n]\setminus i}) - [X_i \triangleright N(X_{[1,n]\setminus i})].$$
$$\nu(X\otimes s) := XN(s) - [X\triangleright N(s)].$$

Remark 5.2.2. Let us explain the idea underlying the following theorem. Recall that we want to prove that there is an R such that $\nu(X\otimes s) = N(R(X\otimes s))$ and $ZR = 0$. In other words, we have to rewrite $\nu(X\otimes Y^n)$ as the N of something, which we are going to try by induction on n. So given the fact that there is an $r(X,Y^n)$ such that
$$\nu(X,Y^n) = N(XY^n + r(X,Y^n)),$$
we want to construct $r(X,Y^{n+1})$ such that it satisfies this equation with n replaced by $n+1$. In the course lemma 5.2.3.1 we arrive at
$$\nu(X,Y^{n+1}) = \nu(Y,XY^n)+N(Yr(X,Y^n)+r(Y,r(X,Y^n))+[Y,X]Y^n+r([Y,X],Y^n)).$$
So at that stage we have expressed $\nu(X,Y^{n+1})$ in terms of the N of something and $\nu(Y,XY^n)$. So it remains to express $\nu(Y,XY^n)$ in terms of $N(..)$ and $\nu(X,Y^{n+1})$ in an independent way, so that we get two equations
$$a\nu(X,Y^{n+1}) + b\nu(Y,XY^n) = N(..),$$
$$b\nu(X,Y^{n+1}) + d\nu(Y,XY^n) = N(..).$$
Which we may solve for $\nu(X,Y^{n+1})$. This second equation is fournished by point 2 of the theorem, which makes the proof possible. The map $R = M + r$ thus defined indeed satisfies $ZR = 0$, by definition.

Theorem 5.2.3. *These maps satisfy the following properties:*
1. $N = \nu S + Z$.
2. $\nu(Y,XY^n) = \frac{-1}{n+1}\nu(X,Y^{n+1}) + N(\frac{n+2}{n+1}XY^{n+1})$.
3. $\nu(X,Y^{n+1}) = Y.\nu(X,Y^n) - [Y\triangleright\nu(X,Y^n)] + \nu([Y,X],Y^n)$.
4. $\nu = N(M+r)$. *(This is the main result.)*

Proof

1. Since $S(1) = 0$, we have $N(1) = (\nu S + Z)(1)$, and for higher degree, we have to check $N(X_1..X_n) = \nu S(X_1..X_n)$, which is true by definition of S, ν and N.

2.

$$N(XY^{n+1}) = \nu S(XY^{n+1}) = \frac{1}{n+2}(\nu(X, Y^{n+1}) + (n+1)\nu(Y, XY^n)),$$

$$\Rightarrow (n+2)N(XY^{n+1}) - \nu(X, Y^{n+1}) = (n+1)\nu(Y, XY^n).$$

3.

$$LHS = XN(Y^{n+1}) - [X \triangleright N(Y^{n+1})]$$

$$= XYN(Y^n) - X[Y \triangleright N(Y^n)] - [X \triangleright YN(Y^n)] + [X \triangleright [Y \triangleright N(Y^n)]]$$

$$= Y\nu(X, Y^n) - [Y \triangleright XN(Y^n)] - [X \triangleright Y]N(Y^n) + [[X, Y] \triangleright N(Y^n)]$$

$$+Y[X \triangleright N(Y^n)] + [Y \triangleright X]N(Y^n) - Y[X \triangleright N(Y^n)] + [Y \triangleright [X \triangleright N(Y^n)]]$$

$$= Y\nu(X, Y^n) - [Y \triangleright \nu(X, Y^n)] + [Y, X]N(Y^n) - [[Y, X] \triangleright N(Y^n)] = RHS.$$

4. We prove that $\nu(X \otimes s) = N(M + r)(X \otimes s)$ by induction on $n = |s|$. $n = 0$: $\nu(X, 1) = X = N(X) = N(X.1 + r(X, 1)) = N(M + r)(X, 1)$. Assume true up to n. Then by polarization it suffices to prove the following

   Lemma 5.2.3.1. $\nu(X, Y^{n+1}) = N(XY^{n+1} + r(X, Y^{n+1}))$.

   Proof

$$\nu(X, Y^{n+1}) = Y\nu(X, Y^n) - [Y \triangleright \nu(X, Y^n)] + \nu([Y, X], Y^n)$$

$$= YN(XY^n + r(X, Y^n)) - [Y \triangleright N(XY^n + r(X, Y^n))]$$

$$+N([Y, X]Y^n + r([Y, X], Y^n))$$

$$= \nu(Y, XY^n) + \nu(Y, r(X, Y^n)) + N([Y, X]Y^n + r([Y, X], Y^n))$$

$$= \nu(Y, XY^n) + N(Yr(X, Y^n) + r(Y, r(X, Y^n)) + [Y, X]Y^n + r([Y, X], Y^n))$$

$$=^{9} \nu(Y, XY^n) + N(\frac{n+2}{n+1}r(X, Y^{n+1}))$$

$$= \frac{-1}{n+1}\nu(X, Y^{n+1}) + N(\frac{n+2}{n+1}XY^{n+1} + \frac{n+2}{n+1}r(X, Y^{n+1}))$$

$$\Rightarrow \nu(X, Y^{n+1}) = N(XY^{n+1} + r(X, Y^{n+1})).$$

$\square$

$\square$

[9]This equality motivates the definiton of r.

5.3. **Applications to subgaussian algebras and boundary terms.** In the rest of this section we will apply some of the above formulae to explicitly construct the inverse of normal ordering in the subgaussian case, to give a generalization of the subgaussian formula for $[X \triangleright N(YZ)]$, proved in theorem 3.1.2, and to construct the solution of the Schwinger-Dyson equation with prescribed boundary term.

Definition 5.3.1. *Given a subgaussian contraction algebra, we define $\bar{N} : Sym(L) \to Sym(L)$, and $\rho : (K \oplus L) \otimes Sym(L) \to Sym(L)$, inductively by: ($r(1, s) := 0$).*

$$\begin{gathered}\bar{N}(1) := 1,\\ \bar{N}(Y^{n+1}) := Y\bar{N}(Y^n) + r(Y, \bar{N}(Y^n)) + \bar{N}([Y \triangleright Y^n]).\\ \rho(1, s) := \rho(X, 1) := 0,\\ \rho(X, Y^{n+1}) := [[Y \triangleright X] \triangleright Y^n] + \rho([Y \triangleright X], Y^n) + Y\rho(X, Y^n)\\ r(Y, \rho(X, Y^n)) + r([X \triangleright Y], Y^n) + r(Y, [X \triangleright Y^n]).\end{gathered}$$

Theorem 5.3.2. *We then have:*

1. $[X \triangleright N(Y^n)] = N([X \triangleright Y^n] + \rho(X, Y^n))$.
2. $N^{-1} = \bar{N}$.

Proof

1. We will prove this identity by induction on n. For $n = 0$ it reads $0 = 0$, so assume it to be true up to n, we will now prove it for $n + 1$.

$$\begin{gathered}[X \triangleright N(Y^{n+1})] - N([X \triangleright Y^{n+1}])\\ = [X \triangleright YN(Y^n)] - [X \triangleright [Y \triangleright N(Y^n)]] - N([X \triangleright Y]Y^n) - N(Y[X \triangleright Y^n])\\ = [X \triangleright Y]N(Y^n)_1 - [[X, Y] \triangleright N(Y^n)]_2 - [X \triangleright Y]N(Y^n)_1 - YN([X \triangleright Y^n])_3\\ +Y[X \triangleright N(Y^n)]_3 - [Y \triangleright [X \triangleright N(Y^n)]]_4 + [[X \triangleright Y] \triangleright N(Y^n)]_2 + [Y \triangleright N([X \triangleright Y^n])]_4\\ +N(r([X \triangleright Y], Y^n)) + N(r(Y, [X \triangleright Y^n]))\\ = [[Y \triangleright X] \triangleright N(Y^n)]_2 + YN(\rho(X, Y^n))_3 - [Y \triangleright N(\rho(X, Y^n))]_4\\ +N(r([X \triangleright Y], Y^n) + r(Y, [X \triangleright Y^n]))\\ = N\{[[Y \triangleright X] \triangleright Y^n] + \rho([Y \triangleright X], Y^n) + Y\rho(X, Y^n) + r(Y, \rho(X, Y^n))\\ +r([X \triangleright Y], Y^n) + r(Y, [X \triangleright Y^n]))\} = N(\rho(X, Y^{n+1})).\end{gathered}$$

2. We will prove by induction on n that $N\bar{N}(Y^n) = \bar{N}N(Y^n) = Y^n$. Indeed, this is true by definition for $n = 0$, so assume the identities hold up to n, then:

$$\begin{gathered}N\bar{N}(Y^{n+1}) = N(Y\bar{N}(Y^n) + r(Y, \bar{N}(Y^n))) + [Y \triangleright Y^n]\\ = \nu(Y, \bar{N}(Y^n)) + [Y \triangleright Y^n]\\ = YN\bar{N}(Y^n) - [Y \triangleright N\bar{N}(Y^n)] + [Y \triangleright Y^n] = Y^{n+1},\end{gathered}$$

Next, to prove $\bar{N}N = id$, we first prove that $\bar{N}$ is surjective. This follows from $\bar{N}(Y^{n+1}) = Y^{n+1} \mod Sym^{[0,n]}(L)$, which in turn follows from the definition, by induction on n. Therefore, for every Y^n there is an s_n such that $Y^n = \bar{N}(s_n)$, so that:

$$\bar{N}N(Y^n) = \bar{N}N\bar{N}(s_n) = \bar{N}(s_n) = Y^n,$$

which proves the identity.

□

Remark 5.3.3. The following theorem is motivated by integration over manifolds with boundary: Suppose we already know integration over the boundary ∂M of a manifold. Then in particular, if μ is a volume form on M we know $\tilde{J} : X \otimes f \mapsto \int_{\partial M} f i_X \mu$. This last association is related to the integral over M through the Schwinger-Dyson equation for $I : f \mapsto \int_M f\mu$:

$$I(\nabla(X)f + X(f)) = \int_M L_X(f\mu) = \int_M di_X(f\mu) = \int_{\partial M} f i_X \mu.$$

In algebraic language, this leads us to consider the equation $I(-Xs + [X \triangleright s]) = \tilde{J}(X \otimes s)$, i.e. setting $J(X \otimes s) := -\tilde{J}(X \otimes N(s))$, we become interested in the solvability of the equation $I(XN(s) - [X \triangleright N(s)]) = J(X \otimes s)$, which is what the following theorem is about:

Theorem 5.3.4. *Setting $\pi := S(M + r)$, the following are equivalent:*

1. $I(XN(s) - [X \triangleright N(s)]) = J(X \otimes s)$,
2. $IN = JS + I(1)Z$ *and* $J = J\pi$.

Proof
First, using theorems 5.1.4 and 5.2.3, with $R := M + r$, we have the following properties: $ZR = 0$, $SZ = 0$, $\nu = \nu SR$, $N = \nu S + Z$. We now prove the theorem. By definition, (1) is equivalent with $I\nu = J$; Next:
$(1) \Rightarrow (2) : IN = I\nu S + IZ = JS + I(1)Z;\ J = I\nu = I\nu SR = JSR = J\pi.$
$(1) \Leftarrow (2) : I\nu = I\nu SR = I(N - Z)R = INR = JSR + I(1)ZR = J\pi + 0 = J.$

□

Remark 5.3.5. If N is invertible, then what we have done is to solve the inhomogeneous linear equation $I\nu = J$ as $I = JSN^{-1} + E(1)ZN^{-1}$, which as expected is of the form $I_p + I_0$, where I_p is any particular solution, and I_0 is the general solution of the homogeneous equation. Further, note that $\pi^2 = \pi$, see theorem 5.1.4.

6. Conclusion and acknowledgements.

The following is a list of the main results of this work.

1. A useful generalization of contractions to non-Gaussian weights S is the second derivative of S, written as a polynomial in the first derivatives. In view of the fact that normal ordering can be defined in terms of contractions, this leads to a non-Gaussian notion of normal ordering.
2. For the Gaussian case, normal ordering is an invertible operation.
3. Invertibility of normal ordering is interesting for non-Gaussian integrals too, since in that case the inverse of normal ordering is directly related to the solution of the Schwinger-Dyson equation.
4. This statement can be generalized to a nonabelian setting.
5. We defined the notion of a subgaussian weight, for which a number of generalized Wick rules can be derived. Examples of subgaussian algebras can be found in two-dimensional conformal field theory.
6. We have avoided the use of operators and Hilbert spaces. The notion of composite operators was replaced by that of a left extension of a contraction. This approach avoids the problems encountered by Johnson and Low, namely that what they call “commutators” of “operators” do not satisfy the Jacobi-identity.
7. A volume form can be algeberaically characterized up to a constant by its divergence. This leads to the possibility of defining a calculus with differential forms of finite codegree on a possibly infinite dimensional manifold, starting from a given divergence. These differential forms are suited to formulate the Schwinger-Dyson equation for infinite dimensional differential forms. (See appendix E).

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Appendix A. Motivations.

A.1. **Motivation for the need of a functional integral.** There are a number of reasons to want to have a good definition of functional integration:

1. They are useful for the compact description of the time averaged behaviour of mechanical systems that consist of a large number of particles, for example a liter of gas.
2. Idem for the compact description of the behaviour of elementary particles.
3. They are useful in geometry, and more particularly in the geometry of low-dimensional manifolds, where other methods seem to fail.

A.2. Motivation for investigating the Schwinger-Dyson equation.

A.2.1. *Comparison of different methods.* A universally applicable definition of functional integration seems to this date not available. Various methods are used to compute functional integrals, each method in itself being a possible definition. Such definitions have various weak points however:

1. It is usually hard or impossible to prove existence of the integral.
2. Methods of calculation are often of limited applicability.

This motivates one of our aims, which is to find a definition of functional integration which does not have these inconvenients. In order to find out where to start, let us have a look at some methods of calculation:

1. Gaussian functional integrals are computed using Wick's rules.
2. For those that can be deformed into a Gaussian integral, we can try defining the integral using a perturbation series. However this often fails because the series does not converge, see remark 2.3.2. And even if it does, then we cannot be sure that it converges to what we are looking for, since when dealing with singular perturbations the dependence on the perturbations may not be analytical.
3. In other special cases, one can make use of special symmetry properties which determine the integral.
4. If the integral has already been defined for certain values of some parameter, then one may use analytic continuation as the definition for other values of that parameter.
5. If the base manifold is cylindrical, i.e. of the form $N \times [0, 1]$, then the path integral may be defined as the exponent of the Hamilton operator, that is if the operator itself is defined.
6. Only two approaches however do not specify in advance which type of integral is to be considered:
   (a) The Schwinger-Dyson equation is an equation that is independent of the type of integral.
   (b) The same holds for the continuum limit of a discretized functional integral.

Thus if we are to follow any of these approaches, it seems that we have to choose between the Schwinger-Dyson approach and the discretized approach. Before making this choice, let us recall some history of usual integration:

1. First, Newton-Leibnitz defined integration as the inverse of differentiation. I.e. given a function f, the integral over $[a, b]$ of f is $F(b)$, where F the unique solution of $F(a) = 0$, $F' = f$, if it exists.

2. The next step by Riemann can be seen either as the existence theorem to the above problem for f Riemann-integrable, or by uniqueness it can be taken to be the definition of the integral of f.
3. When the integral came to be seen as a linear functional, Lebesgue showed that the Riemann-integral could be extended continuously to the larger class of Lebesgue-integrable functions.

What we learn from this historical development is that it is easier to work with defining properties like $F' = f$ rather than with the actual construction of the integral by a limiting process such as in Riemann's approach. This defining property does not in itself refer to any measure. Furthermore, the limiting approach will not be of any help to us for the exact calculation of explicit integrals in terms of known functions, for which we will always use the defining property instead of Riemann sums. Taking this lesson into account for our search for the definition of the functional integral, we will at this point give up the limiting approach: It is probably too ambitious a task to construct the integral before being accustumed to the defining properties. So what remains is the Schwinger-Dyson equation, which as we are suggesting is the analogue of $F' = f$.

A.2.2. *Hints that it is possible.* There are a number of vague reasons to beleive that it makes sense to define a linear functional up to a scalar by the Schwinger-Dyson equation. Here is a list of such reasons:

1. It works if the weight is Gaussian.
2. Next, when perturbing a Gaussian weight S_0 to a nongaussian one $S_0 + \lambda S_1$, the Schwinger-Dyson equation reads:
$$\langle f \partial_i S_0 \rangle = \langle \partial_i f \rangle - \lambda \langle f \partial_i S_1 \rangle.$$
One may try solving this iteratively by starting with the unique solution $\langle . \rangle_0$ at $\lambda = 0$, and continuing with
$$\langle f \partial_i S_0 \rangle_{n+1} = \langle \partial_i f \rangle_{n+1} - \lambda \langle f \partial_i S_1 \rangle_n.$$
Which leads to a unique formal series in λ.
3. Finally, let us give a very rough sketch of an argument independent of the type of weight S used, but using positivity, that leads to the Schwinger-Dyson equation being a defining property apart from the scalar: Assume I is a functional that satisfies the conditions to Riesz' theorem. Then there is a measure ν such that $I(f) = \int f \nu$. "Divide" this measure by μ to get a positive function, and take the logarithm, so as to get: $I(f) = \int f e^{-P} \mu$. Since I now satisfies the Schwinger-Dyson equation for both S and P, we have $\forall_f I(\partial_i (S - P) f) = 0$, which by positivity gives $\partial_i(S) = \partial_i(P)$, or $P = S + c$, so that $I(f) = K. \int f e^{-S} \mu$, so that I is determined up to a positive scalar.

Now each of the above arguments unsatisfactory: Point (1) is a special case. So is point (2), and it only refers to formal power series, and finally point (3) is not a correct proof. Thus we were led to find other ways of proving existence and uniqueness.

A.3. **Motivation for the need of generalized normal ordering.**

A.3.1. *The use of normal ordering to avoid certain divergeces.* Recall that for $D \geq 3$,

$$\int_{\{\phi:\mathbb{R}^D\to\mathbb{R}\}} e^{-\int_{\mathbb{R}^D}\phi\Delta\phi}\phi(x)\phi(y) = \frac{K}{|x-y|^{D-2}}.$$

In particular, we see that this integral is undefined for $x = y$. The divergent limit $x \to y$ is known as an ultraviolet [10] (UV) divergence. UV divergences are not specific to Gaussian integrals, but in the non-Gaussian cases the exact answer is usually not known, so that these divergences are more difficult to illustrate. (UV) divergences prevent one from taking the average of more complicated, functions of ϕ at one point. Examples involving these so-called composite insertions at the point x are $\langle\phi^2(x)\phi(y)\rangle$ and $\langle e^{i\phi(x)}\phi(y)\rangle$.

For Gaussian integrals there is an operation $f \mapsto (: f :)$ called normal ordering, acting on the integrand, which in a way circumvents these UV divergences when working with composite insertions. Let us illustrate this without going into the exact definition of normal ordering. First one may show that for linear functions A_i of ϕ:

$$\langle\prod_{i=1}^n e^{A_i}\rangle = \prod_{i,j} e^{\langle A_iA_j\rangle/2},$$

whereas the normal ordering operation has the property that

$$\langle\prod_{i=1}^n(: e^{A_i} :)\rangle = \prod_{i\neq j} e^{\langle A_iA_j\rangle/2}.$$

Now as we saw, in infinite dimensions, the expression $\langle AB\rangle$ is not well defined for all combinations A and B. Indeed, suppose the integral is over functions $\phi : \mathbb{R}^D \to \mathbb{R}$, and let $A_x(\phi) := \phi(x)$. Then we saw that

$$\langle A_xA_y\rangle = \frac{K}{|x-y|^{D-2}}.$$

So $\langle A_xA_x\rangle$ is undefined. Now $\langle\prod_{i=1}^n \exp(A_{x_i})\rangle$, is undefined because it involves $\langle A_{x_i}A_{x_i}\rangle$, but $\langle\prod_{i=1}^n N(\exp(A_{x_i}))\rangle$ is well defined if the x_i's are different, because the effect of normal ordering is not to include the term $i = j$.

Thus, for Gaussian integrals, one knows how to circumvent ultra violet divergences: By using normal ordering. Now from a higher point of view there is nothing particularly special about Gaussian integrals compared to other integrals exept the fact that they can be explicitly calculated. So we ask the question: Can we find an analogue of the normal ordering procedure for non-Gaussian integrals, so that analogous UV problems can be handled in a similar way?

A.3.2. *Naturality of normal ordering.* Finally, for geometrical applications, it is important to keep the naturality of all constructions in mind. Thus, if the weight S is naturally associated to some geometrical objects, and normal ordering is naturally associated to weights, then normal ordering is naturally associated to geometric objects.

[10] Ultraviolet radiation has shorter wavelength than visible light and is thus associated with the small distance limit $x \to y$ in the base manifold $\mathbb{R}^D$. In the same way one speaks of infrared divergences, i.e. being associated to large distances.

A.4. **Motivation for calling N normal ordering.** The map N is related to what is known as normal ordering of operators. It was initially introduced by Houriet and Kind (1949) to reproduce Feynman's diagrammatic rules for perturbation theory. We will here make the link with what is usually called normal ordering, and show that the so-called canonical quantization procedure is a way to solve the Schwinger-Dyson equation for Gaussian integrals.

A.4.1. *Operator normal ordering.*

Remark A.4.1. The reader who is familiar with operator normal ordering may note the following: Let $J(z) = \oplus_n J_n z^{-n-1}$, with commutation relations $[J_n, J_m] = n\delta_{n+m,0}$, and $J_{n\geq 0}|0\rangle = 0$, $\langle 0|J_{n<0} = 0$. Set $J^+(z) := \oplus_{n\geq 0} z^{-n-1}J_n$, and $J^- := J - J^+$. Then one checks that:

1. $\langle 0|J^-(z) = 0$, $J^+(z)|0\rangle = 0$.
2. $[J_+(\zeta), J_+(z)] = 0$,
3. $[J_+(\zeta), J(z)] = \frac{1}{(\zeta-z)^2}$ for $|z| < |\zeta|$.

Thus, apart from analytic continuations, we have $[J_+(\zeta), J(z)] = [J_+(z), J(\zeta)]$. This motivates the following definition, where W is to be thought of as the algebra of operators, V the "subspace" spanned by the $J(z)$'s, and ϕ, ψ, say, as $J(z), J(\zeta)$:

Definition A.4.2. *A Wick algebra will be a combination (V, W, π), where*

1. *W is an associative, not necessarily commutative algebra with unit, and V is a subvectorspace of W. Thus we have a map $T(V) \to W$. Elements of V will be denoted as ϕ, ψ.*
2. *$\pi : V \to W$; $\phi \mapsto \phi^+$; $\phi^- := \phi - \phi^+$.*

such that with $[w_1, w_2] := w_1w_2 - w_2w_1$:

1. *$[\phi^+, \psi^+] = [\phi^-, \psi^-] = 0$,*
2. *$[\phi^+, \psi^-] \in K \leq W$ (K=scalars times unit),*
3. *$[\phi^+, \psi] = [\psi^+, \phi]$.*

We set $[.\triangleright.] : V \otimes W \to W$; $[\phi \triangleright w] := [\phi^+, w]$. An integral for such an algebra is a map: $\langle 0|.|0\rangle : W \to K$, such that $\langle 0|\phi^- w|0\rangle = \langle 0|w\phi^+|0\rangle = 0$.

Definition A.4.3. *For a Wick algebra, we define a map $T(V) \to W; t \mapsto (: t :)$, inductively by:*

$$(: 1 :) := 1;\ (: s\phi :) := \phi^-(: s :) + (: s :)\phi^+.$$

Theorem A.4.4. *For $s, t \in T(V)$, and $\phi, \psi \in V$:*

1. *$[.\triangleright.]$ is a derivation on the right, and $[\phi \triangleright \psi] = [\phi, \psi^-] = [\phi \triangleright \psi^-]$.*
2. *$(: \phi_1..\phi_n :) = (: \phi_{\sigma(1)}..\phi_{\sigma(n)} :)$.*
3. *$(: \phi s :) = \phi(: s :) - (: [\phi \triangleright s] :)$.*
4. *$(: \phi s :) = \phi(: s :) - [\phi \triangleright (: s :)]$.*
5. *$\langle 0|\phi_1..\phi_n|0\rangle = \langle 0|\phi_{\sigma(1)}..\phi_{\sigma(n)}|0\rangle$.*

Proof

1. We have

$$[\phi \triangleright w_1w_2] = [\phi^+, w_1w_2] = [\phi^+, w_1]w_2 + w_1[\phi^+, w_2] = [\phi \triangleright w_1]w_2 + w_1[\phi \triangleright w_2],$$

and further, $[\phi \triangleright \psi] = [\phi^+, \psi] = [\phi^+, \psi^-] = [\phi, \psi^-]$.

2. It suffices to prove that $(: s\phi t :) = (: st\phi :)$. We first prove this for $|t| = 1$, i.e. we prove $(: s\phi\psi :) = (: s\psi\phi :)$. Indeed:
$$LHS = \psi^-(: s\phi :) + (: s\phi :)\psi^+$$
$$= \psi^-\phi^-(: s :) + \psi^-(: s :)\phi^+ + \phi^-(: s :)\psi^+ + (: s :)\phi^+\psi^+$$
$$= \phi^-\psi^-(: s :) + \phi^-(: s :)\psi^+ + \psi^-(: s :)\phi^+ + (: s :)\psi^+\phi^+ = RHS$$
Next we proceed by induction on $n := |t|$. So let us assume the identity to hold up to $n = |t|$. Then:
$$(: s\phi t\psi :) = \psi^-(: s\phi t :) + (: s\phi t :)\psi^+$$
$$= \psi^-(: st\phi :) + (: st\phi :)\psi^+ = (: st\phi\psi :) = (: st\psi\phi :).$$
3. Induction on $|s|$. $(: \phi 1 :) = \phi = \phi(: 1 :) - (: [\phi \triangleright 1] :)$. Next assume the identity holds for s. We will prove it for $s\psi$:
$$(: \phi s\psi :) = \psi^-(: \phi s :) + (: \phi s :)\psi^+$$
$$= \psi^-(\phi(: s :) - (: [\phi \triangleright s] :)) + (\phi(: s :) - (: [\phi \triangleright s] :))\psi^+$$
$$= \psi^-\phi(: s :) + \phi(: s :)\psi^+ - (: [\phi \triangleright s]\psi :)$$
$$= \phi\psi^-(: s :) + \phi(: s :)\psi^+ - [\phi, \psi^-](: s :) - (: [\phi \triangleright s]\psi :)$$
$$= \phi(: s\psi :) - (: [\phi \triangleright s\psi] :).$$
4. Induction on $|s|$. $(: \phi 1 :) = \phi = \phi(: 1 :) - [\phi \triangleright (: 1 :)]$. Next assume the identity holds for s. We will prove it for $s\psi$.
$$(: \phi s\psi :) = \psi^-(: \phi s :) + (: \phi s :)\psi^+$$
$$= \psi^-(\phi(: s :) - [\phi \triangleright (: s :)]) + (\phi(: s :) - [\phi \triangleright (: s :)])\psi^+$$
$$= \phi(\psi^-(: s :) + (: s :)\psi^+) - [\phi, \psi^-](: s :)$$
$$-\psi^-[\phi \triangleright (: s :)] - [\phi \triangleright (: s :)]\psi^+$$
$$= \phi(: s\psi :) - [\phi \triangleright \psi](: s :) - \psi^-[\phi \triangleright (: s :)] - [\phi \triangleright (: s :)]\psi^+$$
$$= \phi(: s\psi :) - [\phi \triangleright \psi^-(: s :) + (: s :)\psi^+] = \phi(: s\psi :) - [\phi \triangleright (: s\psi :)].$$
5. It suffices to prove that
(a) $\langle 0|\phi s|0\rangle = \langle 0|s\phi|0\rangle$.
(b) $\langle 0|\phi\psi s|0\rangle = \langle 0|\psi\phi s|0\rangle$
Indeed, for the first formula, since $[\phi^+, \psi] = [\psi, \phi^-]$, we have $[\phi^+, s] = [s, \phi^-]$ for $s \in T(V)$, so that: $\langle 0|\phi s|0\rangle = \langle 0|\phi^+ s|0\rangle = \langle 0|[\phi^+, s]|0\rangle = \langle 0|[s, \phi^-]|0\rangle = \langle 0|s\phi^-|0\rangle = \langle 0|s\phi|0\rangle$. For the second formula:
$$\langle 0|\phi\psi s|0\rangle = \langle 0|[\phi \triangleright \psi s]|0\rangle = \langle 0|[\phi \triangleright \psi]s|0\rangle$$
$$+\langle 0|\psi[\phi \triangleright s]|0\rangle = \langle 0|[\psi \triangleright \phi]s|0\rangle + \langle 0|[\psi \triangleright [\phi \triangleright s]]|0\rangle$$
$$= \langle 0|[\psi \triangleright \phi]s|0\rangle + \langle 0|[\phi \triangleright [\psi \triangleright s]]|0\rangle = \langle 0|\psi\phi s|0\rangle.$$

□

Remark A.4.5. The point of the above theorem is the fact that either of the properties 3 or 4 of (::) together with $(:1:) = 1$ are defining properties, using only $[.\triangleright.]$. This version of the definition of normal ordering is very similar to the definition of N, which was $N(1) := 1$ and

$$N(S_{i_0}..S_{i_n}) := S_{i_0}N(S_{i_1}..S_{i_n}) - \partial_{i_0}N(S_{i_1}..S_{i_n}).$$

We see that these two coincide if $S_i \in V$ and $[S_i^+, S_j^-] = g_{ij}$. If this is so we say that the Wick algebra is a canonical quantization of the Gaussian action $S = g_{ij}x^ix^j/2$. We already know that N is related to the solution of the Schwinger-Dyson equation. We will now proceed to make the link between the integrals for a Wick algebra and the solutions of the Schwinger-Dyson equation:

Theorem A.4.6. *Let $\langle 0|.|0\rangle$ be an integral for a Wick algebra (V, W, π). Let Z be the zero projection $T(V) \to K$. Then $\langle 0|(:s:)|0\rangle = Z(s).\langle 0|1|0\rangle$.*

Proof
For $s = 1$ this reads $\langle 0|1|0\rangle = \langle 0|1|0\rangle$. Further,

$$\langle 0|(:s\psi:)|0\rangle = \langle 0|\psi^-(:s:) + (:s:)\psi^+|0\rangle = 0,$$

by definition of an integral.

□

Remark A.4.7. What we have obtained is the following:

1. N is determined using $[\partial_i S \triangleright \partial_j S] := \partial_i\partial_j S$, by $N(s\psi) = \psi N(s) - [\psi \triangleright N(s)]$, and any solution I of the Schwinger-Dyson equation satisfies $IN(s) = Z(s).I(1)$.
2. (::) is determined using $[\phi \triangleright \psi] = [\phi^+, \psi]$, by $(:s\psi:) = \psi(:s:) - [\psi \triangleright (:s:)]$, and any integral $\langle 0|.|0\rangle$ satisfies $\langle 0|(:s:)|0\rangle = Z(s).\langle 0|1|0\rangle$.
3. If a Wick algebra is a canonical quantization of $S = \frac{1}{2}g_{ij}x^ix^j$, then $\partial_i S \in V$ and $[S_i^+, S_j^-] = g_{ij}$.

This amounts to the following: Canonical quantization is a way to solve the Schwinger-Dyson equation for Gaussian integrals. In that case normal ordering (::) is the analogue of the map N that we introduced in the context of not necessarily Gaussian integrals, and this motivates our calling N normal ordering.

A.5. **Motivation for the need of a generalized Wick calculus.** The calculation of commutation relations of normal ordered products like

$$L_n = \frac{1}{2(k + c^\vee)} \sum_{j\in\mathbb{Z}} g_{ab}(:J^a_{-j}J^b_{j+n}:)$$

using mode expansions is cumbersome. Now on the one hand, the special case where the J^a_n's are an abelian Kac-Moody algebra is much easier because it can be calculated using Wick's calculus for Gaussian field theories. But on the other hand on the level of mode expansion calculations, there is not that much difference between the free and the non-free calculation. Thus we are led to try and formulate generalized Wick rules of computation which can be applied to the non-Abelian Kac-Moody case. This is not a new idea in itself: In the field of two-dimensional conformal field theory, it is known how to extend the free Wick calculus to the non-Gaussian conformal case. See for example [24, section 5],[25, appendix A]. I have been looking for an extension of the Wick rules that do not depend on

dimensionality or symmetry. Indeed, I found that replacing for example the free property

$$[a \triangleright (: a_1 a_2 :)] = (: [a \triangleright a_1] a_2 :) + (: a_1 [a \triangleright a_2] :)$$

by

$$[a \triangleright (: a_1 a_2 :)] = (: [a \triangleright a_1] a_2 :) + (: a_1 [a \triangleright a_2] :) + \frac{1}{2}(: [[a_1 \triangleright a] \triangleright a_2] :) + \frac{1}{2}(: [[a_2 \triangleright a] \triangleright a_1] :)$$

gives the right result for the Kac-Moody calculation. One of the aims was to give a basis to such a generalized Wick rule, and to find out more general rules for higher order polynomials. As it turned out this rule could be derived within the setting of subgaussian contractions.

Next, the operator methods used in higher dimensions lead to "operators of which the commutation relations do not satisfy the Jacobi identity". Although a number of scientists seem to find this exciting, I would say it is unacceptable. It seems that the theory of left extensions provides a good alternative to the operator language.

Appendix B. Review of the Gaussian case.

B.1. **Finite and infinite dimensional Gaussian Schwinger-Dyson equations.**

B.1.1. *The finite dimensional case.* The most elementary case of a weight S where the solution is unique by combinatorics is that of Gaussian weights, so let us quickly review it: $S(x) = g(x,x)/2 = g_{ij}x^i x^j/2$, where g is some symmetric bilinear form, not necessarily positive. In that case, since $\partial_i S = g_{ij}x^j$, the Schwinger-Dyson equation reduces to $I(g_{ij}x^j s) = I(\partial_i s)$. If there is a matrix g^{ij} such that $g_{ij}g^{jk} = \delta_i^k$, then $I(x^j s) = g^{jk} I(\partial_k s)$, so that:

$$I(x^{i_0}x^{i_1}...x^{i_n}) = \sum_{j=1}^{n} g^{i_0 i_j} I(x^{i_1}..\hat{x}^{i_j}..x^{i_n}).$$

This is a recurrence relation which determines I completely for polynomials up to $I(1)$. So we see that for Gaussian integrals, the Schwinger-Dyson equation is a defining property. Note by the way that since we have not required g to be positive, we have in a sense defined the expression

$$\frac{\int e^{x^2/2}p(x)dx}{\int e^{x^2/2}dx},$$

for polynomial p, namely as $\langle p \rangle$ where $\langle . \rangle$ is the unique normalized solution to the Schwinger-Dyson equation with weight $S(x) = -x^2/2$. Note however that this solution is not positive, since $\langle x^2 \rangle = -1$. Thus we see that combinatorical integration can be an extension of usual integration. The price we pay is that we lose the normalization constant, because the Schwinger-Dyson equation will never tell us that $\int e^{-x^2}dx = \sqrt{\pi}$; We may also lose positivity, and the solution may not even be unique up to normalization, as we will see in section C.

B.1.2. *Introduction to Gaussian functional integration.* We have now come to the point where we can explain the formula that we gave at the start of this work: The idea is that the Schwinger-Dyson equation is a well defined infinite dimensional equation, which we may try to solve as well. So let us look at an infinite dimensional Gaussian Schwinger-Dyson equation. We will compute the combinatorical integral over the space of differentiable functions $\phi : \mathbb{R}^D \to \mathbb{R}$, with weight $S(\phi) := \int_{\mathbb{R}^D} \partial_i \phi \partial^i \phi/2$, replacing the vectorfields ∂_i that we used before by the generalized vectorfields $\delta_x := \frac{\delta}{\delta\phi(x)}$. So we look for a functional $\langle . \rangle$ that satisfies $\langle \delta_x(S)f \rangle = \langle \delta_x f \rangle$, i.e. with Δ being the Laplace operator:

$$\langle (-\Delta\phi)(x)f \rangle = \langle \delta_x f \rangle.$$

Now we recall from the finite dimensional case that we had to find the inverse matrix g^{ij} such that $g^{ij}g_{jk} = \delta^i_k$ to get to the solution. So let us do the same thing for Δ. [11] The following should always be read distributionally in x, y, z. In

[11] If no boundary conditions are imposed, Δ is not invertible. This matter will be discussed in greater generality in the section on integration over quotient manifolds: The point is that the weight S is invariant under $\phi \mapsto \phi + const.$, and what we are talking about here is the integral over the quotient $\{\phi\}/\sim$ of functions modulo constants. Now $\phi \mapsto \phi(x)$ is not a function on that quotient and this is reflected in the ambiguity in f_D as defined by $\Delta(f_D) = \delta$ (like the addition of a constant): It makes $\langle \phi(x)\phi(y) \rangle$ ill defined, but $\langle \partial_i\phi(x)\partial_j\phi(y) \rangle$ is well defined, which is good since $\phi \mapsto \partial_i\phi(x)$ is a map on the quotient. What is also well defined on the quotient is a product $\prod_{k=1}^{n} e^{ip_k\phi(x_k)}$ if $\sum_k p_k = 0$.

dimension D we have $\Delta(f_D(x)) = \delta(x)$, where $\delta(x)$ is the Dirac delta function, and

$$f_1(x) := \frac{1}{2}|x|;\ f_2(x) := \frac{1}{2\pi}\ln|x|,$$
$$f_{D\geq 3}(x) := \frac{1}{(2-D)Vol(S^{D-1})|x|^{D-2}},$$

Therefore,

$$\langle\phi(x)f\rangle = \int \delta(x-z)\langle\phi(z)f\rangle dz = \int \Delta(f_D(x-z))\langle\phi(z)f\rangle dz$$
$$= \int f_D(x-z)\langle(\Delta\phi)(z)f\rangle dz = \int f_D(x-z)\langle -\delta_z f\rangle dz.$$

In particular

$$\langle\phi(x)\phi(y)\rangle = \int f_D(x-z)\langle -\delta(z-y)\rangle dz = -f_D(x-y).$$

This is the formula that we promised to explain.

B.2. **Gaussian contraction algebras.**

Theorem B.2.1. *Gaussian contraction algebras satisfy the following properties:*

1. *They are Abelian:* $\forall_{X,Y\in L}\ [X,Y] = 0$.
2. $[X \triangleright Y] = \langle XY\rangle$.
3. $[X \triangleright N(s)] = N([X \triangleright s])$.
4. $N(e^X) = e^{X-\langle XX\rangle/2}$. *(As formal power series.)*
5. $\langle e^X\rangle = e^{\langle XX\rangle/2}$.
6. $\langle N(e^X)\phi\rangle = \langle e^X\phi\rangle/\langle e^X\rangle$.
7. $N(e^X)N(e^Y) = N(e^{\langle XY\rangle+X+Y})$.
8. *(Wick,* [7, theorem 2] [12]*).* $N(e^{X_1})..N(e^{X_n}) = N(e^{\sum_i X_i+\sum_{i<j}\langle X_iX_j\rangle})$.
9. $\langle\prod_{i=1}^n N(e^{X_i})\rangle = \prod_{i<j} e^{\langle X_iX_j\rangle}$.
10. *For* $n\geq 1$*:* $\langle Y^{2n-1}\rangle = 0$*, and* $\langle Y^{2n}\rangle = (2n-1)!!\langle Y^2\rangle^n$.

Proof

1. $[X \triangleright Y] - [Y \triangleright X] = [X,Y] \in Sym^0 \cap Sym^1 = \{0\}$.
2. Since $[X \triangleright Y]$ is a number, it is equal to $\langle[X \triangleright Y]\rangle$, which equals $\langle XY\rangle$ by the Schwinger-Dyson equation.
3. Recall that for Abelian contraction algebras, we have $N(Xs) = XN(s) - [X \triangleright N(s)]$, since by definition 5.1.3: Abelian $\Rightarrow r(X,s) = 0$, so that by theorem 5.2.3 point 4: $N(Xs) = \nu(X,s)$. We now proceed by induction on $|s|$; $[X \triangleright N(1)] = [X \triangleright 1] = 0 = N([X \triangleright 1])$. Assume true up to $|t|$. Then:

$$[X \triangleright N(Yt)] - N([X \triangleright Yt])$$
$$= [X \triangleright YN(t)] - [X \triangleright [Y \triangleright N(t)]] - N([X \triangleright Y]t) - N(Y[X \triangleright t])$$
$$= [X \triangleright Y]N(t) - [Y \triangleright [X \triangleright N(t)]] - [X \triangleright Y]N(t) - YN([X \triangleright t])$$
$$+Y[X \triangleright N(t)] \quad + [Y \triangleright N([X \triangleright t])] = 0.$$

[12] Wick's theorem expresses products of normal ordered expressions as normal ordered expressions. One recovers it from the above formulation by replacing X by λX and taking derivatives with respect to λ at $\lambda = 0$.

4. Define $a(\lambda) := N(e^{\lambda X})$, and $b(\lambda) := e^{\lambda X - \lambda^2 \langle XX \rangle/2}$. Then $a(0) = 1 = b(0)$, so it suffices to prove that both a and b satisfy the differential equation $\partial_\lambda f(\lambda) = Xf(\lambda) - [X \triangleright f(\lambda)]$. Indeed:
$$\partial_\lambda a(\lambda) = N(Xe^{\lambda X}) = XN(e^{\lambda X}) - [X \triangleright N(e^{\lambda X})] = Xa(\lambda) - [X \triangleright a(\lambda)],$$
and
$$\begin{aligned}\partial_\lambda b(\lambda) = b(\lambda)(X - \lambda\langle XX \rangle) &= Xb(\lambda) - b(\lambda)[X \triangleright \lambda X - \lambda^2 \langle XX \rangle/2] \\ &= Xb(\lambda) - [X \triangleright b(\lambda)].\end{aligned}$$
5. $1 = \langle N(e^X) \rangle = \langle e^X \rangle / e^{\langle XX \rangle/2}$.
6. $\langle N(e^X)\phi \rangle = \langle e^{X - \langle XX \rangle/2}\phi \rangle$.
7. $$LHS = e^{X - \langle XX \rangle/2} e^{Y - \langle YY \rangle/2} = e^{X+Y-\langle (X+Y)^2 \rangle/2 + \langle XY \rangle} = RHS.$$
8. By induction on n from the previous formula.
9. By taking N of the previous formula since $\langle N(e^{\sum_i X_i}) \rangle = 1$.
10. $\langle 1 \rangle = 1$, $\langle Y \rangle = 0$, and
$$\langle Y^{n+2} \rangle = \langle [Y \triangleright Y^{n+1}] \rangle = (n+1)\langle Y^n [Y \triangleright Y] \rangle = (n+1)\langle Y^n \rangle \langle Y^2 \rangle,$$
so $\langle Y^{2n+1} \rangle = 0$, and $\langle Y^{2n+2} \rangle = (2n+1)\langle Y^{2n} \rangle \langle Y^2 \rangle$, so that $\langle Y^4 \rangle = 3\langle Y^2 \rangle^2$, $\langle Y^6 \rangle = 5.3.\langle Y^2 \rangle^3$, etc.

$\square$

Appendix C. Examples of easy weights: Polynomials in one variable.

This section is meant to give the reader a feeling of what type of solutions the Schwinger-Dyson equation can have if that solution is not unique. None of the statements here are very deep since we will concentrate on polynomial weights $S(x)$ of one real variable only. What seemed worth mentioning is the fact that among the possible solutions of the Schwinger-Dyson equation, there are a number of preferred ones (see def C.1.3) which we call "equilibria", because these solutions have some properties in common with equilibria: For example, using the weight $S(x) = -x^2/2 + x^4/4$ one finds equilibria with $\langle x\rangle = \pm 1$, which are the values where S has a minimum. We don't claim that these two things are really the same, but the similarities seemed suggestive enough to use the word equilibrium.

Polynomial weights are particulary illustrative, because both the solution of the Schwinger-Dyson equation and the positivity condition can be automated by computer algebra.

C.1. **Algorithmic solution and positivity of the Schwinger-Dyson equation.**

C.1.1. *An algorithm for the solution of the Schwinger-Dyson equation.* Let $S(x)$ be a polynomial of degree $D+1$, and let $s := \partial S$. The Schwinger-Dyson equation for polynomial integrands p reads $\langle sp\rangle = \langle \partial p\rangle$. One way to transform this equation into a recurrence relation is to write s as $s(x) = \sigma x^D + \rho(x)$, so that:

$$\langle x^D p\rangle = \sigma^{-1}\langle \partial p - \rho p\rangle,$$

This is a recurrence relation, which will fix $\langle . \rangle$ once $\langle x\rangle, \langle x^2\rangle, ..., \langle x^{D-1}\rangle$ are known. The Schwinger-Dyson equation does not determine these values, so that the (normalized) solutions of the Schwinger-Dyson equation form a $(D-1)-$parameter family. Equivalently, the Schwinger-Dyson equation in differential form, see footnote 1, is of order D, so we get D integration constants parametrizing the solutions, of which we substract one for normalization.

A maple procedure that will determine the expectation value $\langle p\rangle$ for $s = \partial S$ in terms of $y_i := \langle x^i\rangle$ for low i is the following:

```
Ex:=proc(p:algebraic,s:algebraic,x:name,y:name)
local r,q,i:
r:=rem(p,s,x,'q'):
if q=0 then
subs({seq(x^i=y.i,i=1..degree(r,x))},r):
else Ex(p,s,x,y):=simplify(Ex(diff(q,x),s,x,y)+Ex(r,s,x,y))
fi:
end:
```

The point is to write p as $p = sq+r$, where the degree of r is lower than that of s. In that case sq can be replaced by ∂q, and the expectation value of r is determined by replacing x^i by y_i.

C.1.2. *An algorithm to generate the positivity conditions.* We will generate a countable number of conditions which will ensure that for all polynomials $p > 0$, we have $\langle p\rangle > 0$. To that end we inductively define the following bilinear symmetric forms on polynomials, depending on $\langle . \rangle$:

$$g_0(p, q) := \langle pq\rangle; g_{n+1}(p, q) := g_n(1, 1)g_n(xp, xq) - g_n(xp, 1)g_n(xq, 1).$$

Theorem C.1.1. *Let $\langle . \rangle$ be a linear form on polynomials. Then we have the following equivalence:*

$$\{\forall_{p>0}\ \langle p \rangle > 0\} \Leftrightarrow \{\forall_n\ g_n(1,1) > 0\}.$$

Proof
First, every positive real polynomial is a sum of squares of nonzero polynomials: Indeed, such a polynomial does not have any real roots, and its complex roots come in conjugate pairs, so that it can be written as $\prod_i (x - a_i)(x - \bar{a}_i)$. Now $(x - a_i)(x - \bar{a}_i)$ is itself a positive polynomial, so it suffices to prove that it can be written as a sum of nonzero squares. Indeed, any second order polynomial is of the form $(x - b)^2 + c$, and this is positive iff $c > 0$, so that $c = d^2$.

Thus, to prove positivity of $\langle . \rangle$, it suffices to prove that $\forall_{p \neq 0}\ \langle pp \rangle > 0$, i.e. $\forall_{p \neq 0}\ g_0(p,p) > 0$. Let us now prove the following statement:

$$g_n(1,1) > 0 \Rightarrow [\{\forall_{p:|p|=k+1, p\neq 0}\ g_n(p,p) > 0\} \Leftrightarrow \{\forall_{p:|p|=k, p\neq 0}\ g_{n+1}(p,p) > 0\}].$$

Indeed, assume that $g_n(1,1) > 0$, then we have to prove that $[A \Leftrightarrow B]$:

$$A \Leftrightarrow \forall_{p\neq 0:|p|=k, b\in\mathbb{R}}\ 0 < g_n(xp+b, xp+b) = b^2 g_n(1,1) + 2b g_n(xp,1) + g_n(xp,xp).$$

The last statement is equivalent to the discriminant being smaller than zero:

$$0 > (2g_n(xp,1))^2 - 4g_n(xp,xp)g_n(1,1) = -4g_{n+1}(p,p) \Leftrightarrow g_{n+1}(p,p) > 0 \Leftrightarrow B.$$

With this result, we can prove the theorem:

($\Rightarrow$). First assume that $\langle p \rangle$ is positive. Then $g_0(1,1) > 0$. Therefore, using the above implication, we have $\forall_{p\neq 0}\ g_0(p,p) > 0 \Leftrightarrow \forall_{p\neq 0}\ g_1(p,p) > 0$, so that $g_1(1,1) > 0$. This in turn allows us to use the implication again, so we get $g_2(p,p) > 0$, etc.

($\Leftarrow$). Now assume that $\forall_n g_n(1,1) > 0$. Then we have $\forall_n$:

$$\{\forall_{p:|p|=k+1, p\neq 0}\ g_n(p,p) > 0\} \Leftrightarrow \{\forall_{p:|p|=k, p\neq 0}\ g_{n+1}(p,p) > 0\}.$$

Thus we may deduce that

$$\forall_k\ g_k(1,1) > 0 \Leftrightarrow \forall_{k, p\neq 0:|p|=0}\ g_k(p,p) > 0 \Leftrightarrow$$

$$\forall_{k, p\neq 0:|p|=k}\ g_0(p,p) > 0 \Leftrightarrow \forall_{p\neq 0}\ g_0(p,p) > 0.$$

$\square$

Remark C.1.2. For example, we have

$$g_0(1,1) = \langle 1 \rangle (:= 1),$$

$$g_1(1,1) = \langle x^2 \rangle - \langle x \rangle^2,$$

$$g_2(1,1) = (\langle x^2 \rangle - \langle x \rangle^2)(\langle x^4 \rangle - \langle x^2 \rangle^2) - (\langle x^3 \rangle - \langle x^2 \rangle \langle x \rangle)^2.$$

Definition C.1.3. *By an equilibrium for the weight S, we mean a solution $\langle . \rangle$ of the Schwinger-Dyson equation for S, which satisfies $\forall_n g_n(1,1) \geq 0$.*

Remark C.1.4. Note that $g_n(1,1)$ is allowed to be zero. In particular, an equilibrium need not be a positive solution of the Schwinger-Dyson equation.

The g_n's can be computed in Maple using the following procedure, which computes $g_n(p,q)$ for the derived weight $s = \partial S$:

```
DEx:=proc(n:nonnegint,p,q,s:algebraic,x:name,y:name)
if n=0 then Ex(p*q,s,x,y)
else DEx(n,p,q,s,x,y):=
DEx(n-1,1,1,s,x,y)*DEx(n-1,x*p,x*q,s,x,y)
-DEx(n-1,x*p,1,s,x,y)*DEx(n-1,x*q,1,s,x,y)
fi:
end:
```

Definition C.1.5. *By a null-equilibrium, we mean an equilibrium such that* $\exists_N \forall_{n \geq N} g_n(1,1) = 0$.

Theorem C.1.6. *(The following are useful to prove that an equilibrium is a null-equilibrium:)*

1. $g_n(1,1) = 0 \Rightarrow \{g_{n+1}(1,1) = 0 \Leftrightarrow g_n(x,1) = 0\}$.
2. $\{g_n(1,1) = g_n(x,1) = 0\} \Rightarrow g_{n+1}(p,1) = 0$.

Proof

1. $$g_{n+1}(1,1) = g_n(1,1)g_n(x,x) - g_n(x,1)g_n(x,1) = -g_n(x,1)^2.$$
2. $$g_{n+1}(p,1) = g_n(1,1)g_n(xp,x) - g_n(xp,1)g_n(x,1) = 0.$$

□

C.2. **Third order and fourth order weights.**

Theorem C.2.1. *Using the above procedures for calculations, one can deduce:*

1. $S(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3$ *has no equilibria.*
2. $S(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ *has multiple equilibria, of which we list a few:*

Name of eq.	$\langle x \rangle$	$\langle x^2 \rangle$	$\langle x^3 \rangle$	$\langle x^4 \rangle$	$g_1(1,1)$	g_2	g_3
1_r	1	1	1	2	0	0	
1_l	-1	1	-1	2	0	0	
1_{mid}	0	0	0	1	0	0	
2_{mid}	0	$\frac{1}{2}(1+\sqrt{5})$	0	$\frac{1}{2}(3+\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	0	0
..	..	..	..	..	..	..	..
∞	0	$\int x^2 e^{-S} dx.$	0	$\int x^4 e^{-S} dx$	> 0	> 0	

3. *There is no equilibrium for* $S(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ *with* $\langle x \rangle = \frac{1}{2}$.

Proof

1. As an illustration of the use of maple, one may type:

```
s:=x+x^2;
ic1:=a;
for i from 1 to 3 do
expect.i:= simplify(Ex(x^i,s,x,ic));
eq.i:=simplify(DEx(i,1,1,s,x,ic));
evalf(solve(eq.i,a));
od;
```

This will tell us that the first two positivity equations for the undetermined integration constant $\langle x\rangle = a$ read:

$$-a^2 - a \geq 0; \;\; -2a^3 - 3a^2 - a - 1 \geq 0.$$

These conditions are equivalent to approximately $a \in [-1, 0]$, and $a \leq -1.4$, which is impossible, so that this weight has no equilibria.

2. For this weight there are two integration constants, $a := \langle x\rangle$ and $b := \langle x^2\rangle$. In terms of these one finds

$$g_1(1,1) = b - a^2,$$
$$g_2(1,1) = b + b^2 - b^3 - 2a^2 + a^2 b.$$

Intersecting the zero's of g_1 and g_2 in the (a, b)-plane gives the points $(-1, 1), (0, 0), (1, 1)$. Therefore, by theorem C.1.6, $g_{n\geq 1}(1, 1) = 0$ at those points, so that these are null equilibria. Other points may be found by intersecting g_n with g_{n+1}. Further, there is obviously a positive equilibrium, corresponding to usual integration: $\langle f\rangle := \int e^{-S} f dx$.

3. Assume $\langle x\rangle = \frac{1}{2}$, and set $b := \langle x^2\rangle$. Then the first three conditions read:

$$g_1 = b - \frac{1}{4} \geq 0,$$
$$g_2 = \frac{5}{4}b + b^2 - b^3 - \frac{1}{2} \geq 0,$$
$$g_3 = \frac{11}{4}b^3 - \frac{107}{16}b^2 + \frac{35}{16}b + \frac{19}{4}b^4 - 3b^5 - \frac{3}{16} \geq 0.$$

These are incompatible. (By just plotting the graphs with high enough resolution.)

□

Appendix D. Non-polynomial contractions.

D.0.1. *More general relations.* Let us drop the topic of uniqueness of solutions of the Schwinger-Dyson equation for a moment, and consider for example the case in which $\partial^2 S$ cannot be written as a function of ∂S; This is the case for example for $S(x) = x^3$. Let us first recall

$$\langle \partial_{i_1}(S)...\partial_{i_n}(S)\rangle = \sum_{k=2}^{n} \langle \partial_{i_2}(S)..\partial_{i_1}\partial_{i_k}(S)..\partial_{i_n}(S)\rangle.$$

Now if $\partial^2 S$ cannot be written in terms of first derivatives, then we have an additional independent equation

$$\langle \partial_{i_1}(S)...\partial_{i_n}(S)\partial_l\partial_m(S)\rangle = \sum_{k=2}^{n} \langle \partial_{i_2}(S)..\partial_{i_1}\partial_{i_k}(S)..\partial_{i_n}(S)\partial_l\partial_m(S)\rangle$$
$$+\langle \partial_{i_2}(S)...\partial_{i_n}(S)\partial_{i_1}\partial_l\partial_m(S)\rangle,$$

and analogously for higher powers in $\partial^2 S$. If $\partial^3 S$ can be written as a polynomial in lower degree derivatives, then there are no more equations to be considered, otherwise we may go on. So, recalling that $\langle . \rangle$ was defined on the algebra $\mathcal{S}$ generated by all derivatives of S, we see that that the only things that are relevant are the algebraic relations that exist between the various derivatives of S. Contraction algebras are objects in which the concept of a weight S has been repaced by relations.

The universal contraction algebra is the algebra in which there are no relations at all. It is just the algebra of formal combinations like

$$(\partial_i\partial_j)(\partial_k\partial_l\partial_m), \textit{ or } [\partial_i \triangleright \partial_j][\partial_k \triangleright [\partial_l \triangleright \partial_m]]$$

which when given a weight S are mapped to $(\partial_i\partial_j S)(\partial_k\partial_l\partial_m S)$; The kernel of this map determines relations in the universal contraction algebra, and so what we mean by a contraction algebra is the universal contraction algebra subject to a number of relations. So a contraction algebra is an abstraction of a weight. The introduction of contraction algebras serves a number of purposes:

1. They save space, because XYZ is shorter than $(XS)(YS)(ZS)$.
2. Some constructions can be made even for the universal contraction algebra, and are therefore weight-independent.
3. For applications to the infinite dimensional case, it is useful to be able to manipulate the expression $(\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x)})$ in the universal algebra, even if its actual evaluation $(\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x)}S)$ is undefined.
4. It may be easier to specify relations than to explicitly give a weight that actually satisfies these relations.

D.0.2. *The noncommutative case, a number of definitions.*

Remark D.0.2. Non-commuting vectorfields lead to a number of complications:

1. We have to take all commutators into account.
2. Like in remark 3.0.3, it is better to talk about divergences ∇ than about weights S.
3. We will also want to consider combinations of the form $(X_1X_2)(Y_1Y_2Y_3)$, which again are to be thought of as $(X_1X_2S)(Y_1Y_2Y_3S)$, or rather when given a divergence as $(-X_1\nabla(X_2))(-Y_1Y_2\nabla(Y_3))$. Since in any case we will have $XY(S) - YX(S) = [X,Y](S)$, we lose no information when dividing out

these expressions by the relation $XY - YX = [X, Y]$. This will also be true for any divergence, because they always satisfy $\nabla([X, Y]) = X(\nabla(Y)) - Y(\nabla(X))$, see theorem E.1.4. In other words, the expressions like (X_1X_2) and $(Y_1Y_2Y_3)$ between the brackets are in the universal enveloping algebra of a Lie algebra, and since we want to multiply these expressions in turn, we end up defining the universal contraction algebra as $Sym(UEA(L))$, the symmetric algebra of the universal enveloping algebra of a Lie algebra L. Finally, if $\tilde{I}$ satisfies the Schwinger-Dyson equation, we may as well consider its pullback to $Sym(UEA(L))$:

$$I((X_1X_2)(Y_1Y_2Y_3)) := \tilde{I}((X_1X_2S)(Y_1Y_2Y_3S)).$$

For I the Schwinger-Dyson equation now reads $I(Xs) = I([X \triangleright s])$, using the notation introduced below:

Definition D.0.3. *Let $UEA(L)$ denote the universal enveloping algebra of L. We define $UEC(L) := Sym(UEA'(L))$.* [13] *We denote the left-multiplicative action of an element $X \in L$ on $UEA(L)$ by $[X \triangleright .]$, and idem for the induced action by derivations on $UEC(L)$. This being so, we will not use the round brackets $(.)$ in UEC any more, and prefer to write $[Y \triangleright Z]Z^2[X \triangleright [Y \triangleright X]] \in UEC(L)$ instead of $(YZ)(Z)(Z)(XYX)$. $[X \triangleright Y]$ is referred to as the contraction of X and Y.*

An ideal in $UEC(L)$ is a subspace $I \leq UEC(L)$ such that $I.UEC(L) \subset I$ and $[L \triangleright I] \subset I$. A contraction algebra is the quotient $UEC(L)/I$ by some ideal. The intersection of a collection of ideals is again an ideal. A contraction algebra can be specified by giving relations, i.e. a subset $R \subset UEC(L)$; In that case it is understood that the quotient is taken by the smallest ideal containing R.

Definition D.0.4. *Equivalently: A contraction algebra is a combination (L, A, ∇), where L is a Lie algebra, A is an associative commutative algebra with unit, L is represented on A by derivations, and $\nabla : L \to A$, such that:*

1. $\nabla([X, Y]) = X(\nabla(Y)) - Y(\nabla(X))$,
2. *The algebra homomorphism $\nabla : UEC(L) \to A$ defined by*
$$[X_1 \triangleright [..[X_{n-1} \triangleright X_n]..] \mapsto X_1..X_{n-1}(-\nabla(X_n))$$
is surjective.

The relation with the previous definition being $I := Ker(\nabla) \leq UEC(L)$, and $A := UEC(L)/I$.

Remark D.0.5. In the main body of the text we have restricted our attention to polynomial contractions. In terms of the above defintion, this means that $A = Sym(L)$, and that ∇ is just the inclusion $L \to Sym(L)$. Relations of polynomial type defined in 3.0.4 induce a map $UEC(L) \to Sym(L)$.

[13] We write UEA' instead of just UEA to indicate that we do not include multiples of the unit in UEA, which would lead to confusions when combined with the unit in Sym. Note that when given a weight S the unit in UEA would get mapped to S, not to 1. We will not keep writing that prime though.

Appendix E. More complicated Schwinger-Dyson equations.

E.1. **The Schwinger-Dyson equation for differential forms.** In this section we will be concerned with generalizing the weight $e^{-S}dx^1..dx^n$ to any volumeform. Since we have the infinite dimensional case in mind, we will avoid talking about volumeforms, and instead talk about divergences ∇. This is enough for our purpose since the Schwinger-Dyson equation only depends on ∇, not on the whole μ. More generally, we will look for a formulation of the Schwinger-Dyson equation for "integration of forms" of finite codegree, applicable in infinite dimensions. Therefore we will first concentrate our efforts on defining forms of finite codegree in possibly infinite dimensions. Only after that will we look for the notion of integration of these objects. We will also consider integrals with prescribed boundary terms.

Definition E.1.1. *By an infinitesimal calculus (A, L), we mean an associative symmetric algebra A with unit, a Lie algebra L, a representation $(f, X) \mapsto fX$ of A on L and a representation $(X, f) \mapsto Xf$ of L on A by derivations, such that: $(gX)f = g(Xf)$ and $[X, fY] = f[X, Y] + (Xf)Y$. By $\Omega(L, A)$ we mean antisymmetric A-linear forms on L with values in A.*

Remark E.1.2. To every manifold is associated an infinitesimal calculus, by letting A be the real functions, and L the vectorfields.

Definition E.1.3. *A volume manifold is a combination (M, μ), where M is a manifold and μ is a volume-form, i.e. a differential form of maximal degree such that μ_m is nonzero in every point $m \in M$. In addition to the infinitesimal calculus (A, L) associated to M alone, we may define $\nabla : L \to A$ by the defining property $\nabla(X)\mu := L_X\mu$.*

Theorem E.1.4. *∇ above satisfies the following properties:*
1. *It is closed: $\nabla([X, Y]) = X(\nabla(Y)) - Y(\nabla(X))$.*
2. *It is local: $\nabla(fX) = X(f) + f\nabla(X)$.*
3. *∇ fixes μ up to multiplication by a locally constant function.*

Proof

1. We have
$$\begin{aligned}\nabla([X, Y])\mu = L_{[X,Y]}\mu = L_X L_Y \mu - L_Y L_X \mu = L_X(\nabla(Y)\mu) - L_Y(\nabla(X)\mu)\\ = X(\nabla(Y))\mu + \nabla(Y)\nabla(X)\mu - Y(\nabla(X))\mu - \nabla(X)\nabla(Y)\mu\\ = (X(\nabla(Y)) - Y(\nabla(X)))\mu.\end{aligned}$$
2. $$\begin{aligned}\nabla(fX)\mu = (i_{fX}d + di_{fX})\mu = d(fi_X\mu) = df \wedge i_X\mu + fdi_X\mu\\ = -i_X(df \wedge \mu) + (i_X df) \wedge \mu + fL_X\mu = (X(f) + f\nabla(X))\mu.\end{aligned}$$
3. Finally, let ν be another volume form giving the same ∇. Since volume forms are proportional, there is a function f such that $\nu = f\mu$. Thus we have
$$X(f)\mu = L_X(f\mu) - fL_X\mu = L_X\nu - f\nabla(X)\mu = \nabla(X)\nu - \nabla(X)\nu = 0.$$
So $X(f) = 0$, i.e. f is locally constant.

□

Remark E.1.5. We might roughly state the above as follows: If we are not interested in a particular normalization of the integral, then all the information contained in (M, μ) is in (A, L, ∇).

Definition E.1.6. *This motivates the definition of a formal volume manifold:*

1. *By a formal volume manifold, we mean a combination (A, L, ∇), where (A, L) is an infinitesimal calculus and ∇ is a divergence for (A, L), meaning that $\nabla : L \to A$, such that:*
   (a) $\nabla([X, Y]) = X(\nabla(Y)) - Y(\nabla(X))$.
   (b) $\nabla(fX) = X(f) + f\nabla(X)$.
2. *Next for any infinitesimal calculus, and $k \in \mathbb{Z}$, we set*

$$\bar{\mathcal{I}}^{-k}(L, A) := \bigoplus_{n \in \mathbb{N}} \Omega^{n-k}(L, A) \otimes_A \bigwedge\nolimits_A^n(L); \;\; \bar{\mathcal{I}} := \bigoplus_{k \in \mathbb{Z}} \bar{\mathcal{I}}^k.$$

Remark E.1.7. Note that the combination $(L, A := Sym(UEA(L)))$ that we saw before has properties similar to formal volume manifolds if we let ∇ just be the inclusion. It is not exactly a formal volume manifold though, since A does not act on L. Next, just as there is a theory of integration over volume manifolds, we will now look for a theory of integration over formal volume manifolds. We will start by formalizing the integrands first, and use $\bar{\mathcal{I}}^{-k}$ to model the notion of a differential form of codegree k, in view of the map below:

Definition E.1.8. *Indeed, suppose that (A, L) comes from (M, μ), then we define*

$$\bar{G} : \bar{\mathcal{I}}^{-k} \to \Omega^{|M|-k}(M); \omega \otimes X_1 ... X_n \mapsto \omega \wedge i_{X_1} ... i_{X_n} \mu.$$

Remark E.1.9. The Cartan calculus of Lie derivation L_X, interior product i_X, and exterior derivation d, (see appendix F), present in (M, μ) can be largely transported to $\bar{\mathcal{I}}(A, L)$, as we will see in the next theorem. This motivates the following definition:

Definition E.1.10. *Let (L, A) be an infinitesimal calculus. Then $Car(L)$ is naturally represented on $\bar{\mathcal{I}}(A, L)$, by the supertensorproduct representation of those on $\Omega(L, A)$ and $\bigwedge(L)$ seperately. (See appendix F). Denote this tensor representation by $\tilde{i}_X, \tilde{L}_X, \tilde{d}$. Given a divergence ∇ for (A, L), we define another map $Car(L) \to End(\bar{\mathcal{I}})$; (It is not a representation):*

1. $i_X := \tilde{i}_X$.
2. $L_X := \tilde{L}_X + M_{\nabla(X)}$, *where M_f denotes multiplication with f.*
3. $d := \tilde{d} + \delta$, *where*

$$\delta(\omega X_1 ... X_n) := (-1)^{|\omega|} \sum_{i=1}^{n} (-1)^{i+1} \nabla(X_i) \omega X_{[1,n] \setminus i}.$$

Remark E.1.11. In general the operators L_X, i_X, d above do not form a representation of the Cartan algebra. They do however modulo elements of the form $i_{X_1}..i_{X_n}\phi$, where ϕ is an element of positive degree, as we will see in a moment. Since $\bar{G}$ annihilates these elements anyway, we lose nothing of interest when taking the quotient by the subspace $\mathcal{O}$ of these "overflow" elements:

Definition E.1.12. *Let (A, L) be an infinitesimal calculus. Then we define:*

1. *The (graded) subspace of overflow forms: $\mathcal{O} \leq \bar{\mathcal{I}}(A, L)$,as the space generated by elements of the form $i_{X_1}..i_{X_n}\phi$, where $|\phi| > 0$ and $n \geq 0$.*
2. *Next, we set $\mathcal{I}(A, L) := \bar{\mathcal{I}}(A, L)/\mathcal{O}$, with induced grading.*
3. *Finally, if (A, L) comes from (M, μ), then we define the map $G : \mathcal{I}(A, L) \to \Omega(M)$ to be the one induced by $\bar{G}$.*

Corollary E.1.13. *Thus, we have a number of identities in $\mathcal{I}(A, L)$, e.g.:*

1. *$\mathcal{I}^{>0}(A, L) = 0$: This is the case $i_{X_1}...i_{X_n}\phi = 0$ with $n = 0$.*
2. *$X(g)f \otimes 1 = (dg)f \otimes X$: $0 = i_X(fdg \otimes 1) = fX(g) \otimes 1 - fdg \otimes X$.*

Theorem E.1.14. *If (L, A, ∇) comes from (M, μ), then:*

1. *$\bar{G}i_X = i_X\bar{G}$.*
2. *$\bar{G}L_X = L_X\bar{G}$.*
3. *$\bar{G}d = d\bar{G}$.*
4. *For any (L, A, ∇), the operators L_X, i_X, d descend to a representation of the Cartan algebra on $\mathcal{I}(A, L)$.*

Proof

1.
$$\begin{aligned} i_X\bar{G}(\omega X_1...X_n) &= i_X(\omega i_{X_1}...i_{X_n}\mu) \\ &= (i_X\omega)i_{X_1}..i_{X_n}\mu + (-1)^{|\omega|}\omega i_X i_{X_1}..i_{X_n}\mu \\ &= \bar{G}((i_X\omega)X_1...X_n + (-1)^{|\omega|}\omega X X_1..X_n) = \bar{G}i_X(\omega X_1...X_n). \end{aligned}$$

2.
$$\begin{aligned} L_X\bar{G}(\omega X_1..X_n) &= L_X(\omega i_{X_1}...i_{X_n}\mu) \\ &= (L_X\omega)i_{X_1}...i_{X_n}\mu + \omega\sum_{j=1}^{n} i_{X_1}..[L_X, i_{X_j}]..i_{X_n}\mu + \omega i_{X_1}..i_{X_n}\nabla(X)\mu \\ &= \bar{G}(\tilde{L}_X(\omega X_1..X_n) + \nabla(X)\omega X_1..X_n) = \bar{G}L_X(\omega X_1..X_n). \end{aligned}$$

3.
$$\begin{aligned} d\bar{G}(\omega X_1..X_n) &= d(\omega i_{X_1}..i_{X_n}\mu) \\ &= (d\omega)i_{X_1}..i_{X_n}\mu + (-1)^{|\omega|}\sum_{j=1}^{n}(-1)^{j+1}\omega i_{X_1}..([d, i_{X_j}] = L_{X_j})..i_{X_n}\mu \\ &= (d\omega)i_{X_1}..i_{X_n}\mu + (-1)^{|\omega|}\sum_{j=1}^{n}(-1)^{j+1}\nabla(X_j)\omega i_{X_1}...i_{X_n}\mu \\ &+(-1)^{|\omega|}\sum_{j=1}^{n}(-1)^{j+1}\omega i_{X_1}..\hat{i}_{X_j}\sum_{k>j} i_{X_{j+1}}..([L_{X_j}, i_{X_k}] = i_{[X_j,X_k]})..i_{X_n}\mu \\ &= \bar{G}((d\omega)X_1..X_n + \delta(\omega X_1..X_n) + (-1)^{|\omega|}\omega\sum_{j<k}(-1)^{j+k+1}[X_j, X_k]X_{[1,n]\setminus ij}). \end{aligned}$$

4. First, using the fact that $\tilde{i}_X, \tilde{L}_X, \tilde{d}$ form a representation, we see that a number of commutators are already correct in $\bar{\mathcal{I}}$:
   (a) $[i_X, i_Y] = [\tilde{i}_X, \tilde{i}_Y] = 0$.
   (b) $[L_X, i_Y] = [\tilde{L}_X, \tilde{i}_Y] + [M_{\nabla(X)}, \tilde{i}_Y] = \tilde{i}_{[X,Y]} + 0 = i_{[X,Y]}$.

(c) $[d, i_Y] - L_Y = [\tilde{d} + \delta, \tilde{i}_Y] - \tilde{L}_Y - M_{\nabla(Y)} = [\delta, i_Y] - M_{\nabla(Y)} = 0$:

$$([\delta, i_Y] - M_{\nabla(Y)})(\omega X_1..X_n) = (\delta i_Y + i_Y \delta)(\omega X_1..X_n) - \nabla(Y)\omega X_1..X_n$$

$$= \delta((i_Y\omega)X_1..X_n + (-1)^{|\omega|}\omega Y X_1..X_n) - \nabla(Y)\omega X_1..X_n$$

$$+ i_Y(-1)^{|\omega|}\sum_{j=1}^{n}(-1)^{j+1}\nabla(X_j)\omega X_{[1,n]\setminus j}$$

$$= (-1)^{|\omega|-1}\sum_{j=1}^{n}(-1)^{j+1}\nabla(X_j)(i_Y\omega)X_{[1,n]\setminus j}$$

$$+\nabla(Y)\omega X_1..X_n - \sum_{j=1}^{n}(-1)^{j+1}\nabla(X_j)\omega Y X_{[1,n]\setminus j} - \nabla(Y)\omega X_1..X_n$$

$$+\sum_{j=1}^{n}(-1)^{j+1}\{(-1)^{|\omega|}\nabla(X_j)(i_Y\omega)X_{[1,n]\setminus j} + \nabla(X_j)\omega Y X_{[1,n]\setminus j}\} = 0.$$

(d)

$$[L_X, L_Y] = [\tilde{L}_X + M_{\nabla(X)}, \tilde{L}_Y + M_{\nabla(Y)}]$$

$$= [\tilde{L}_X, \tilde{L}_Y] + [\tilde{L}_X, M_{\nabla(Y)}] - [\tilde{L}_Y, M_\nabla(X)]$$

$$= \tilde{L}_{[X,Y]} + M_{X(\nabla(Y))} - M_{Y(\nabla(X))} = \tilde{L}_{[X,Y]} + M_{\nabla([X,Y])} = L_{[X,Y]}.$$

Applying the operators L_X, i_X, d to the expression $i_{X_1}..i_{X_n}\phi$, and using the above commutation relations proves that $\mathcal{O}$ is invariant, so that the operators descend to $\mathcal{I}(A, L) = \bar{\mathcal{I}}(A, L)/\mathcal{O}$. So it remains to prove that the following commutators are zero modulo $\mathcal{O}$: $R_X := [L_X, d]$; $\Delta := [d, d]$. First, by lemma F.0.15.1, $[i_X, R_Y] = 0$, and $[i_X, \Delta] = 2R_X$. Using this, we will now prove by induction on n that Δ and R_X are zero mod $\mathcal{O}$ on $\omega X_1..X_n$. Indeed, consider the case $n = 0$ first:

(a)

$$R_X(\omega \otimes 1) = (L_X d - dL_X)(\omega \otimes 1)$$

$$= L_X(d\omega \otimes 1) - d(L_X\omega \otimes 1 + \nabla(X)\omega \otimes 1)$$

$$= L_X d\omega \otimes 1 + \nabla(X)d\omega \otimes 1$$

$$-dL_X\omega \otimes 1 - d(\nabla(X)) \wedge \omega \otimes 1 - \nabla(X)d\omega \otimes 1 \in \mathcal{O}.$$

(b)

$$\Delta(\omega \otimes 1) = 2d^2(\omega \otimes 1) = 2(d^2\omega \otimes 1) = 0.$$

So the statement is true for $n = 0$. Assume it has been proved for $\omega X_1..X_n$, and let A be any linear combination of the operators R_X and Δ. We have to prove that $A(\omega Y X_1..X_n) \in \mathcal{O}$. Using that $[A, i_Y] = A'$, where A' is also of that form, we have:

$$(-1)^{|\omega|}A(\omega Y X_1..X_n) = A(i_Y(\omega X_1..X_n) - (i_Y\omega)X_1..X_n)$$

$$= [A, i_Y](\omega X_1..X_n) \pm i_Y A(\omega X_1..X_n) - A((i_Y\omega)X_1..X_n)$$

$$= A'(\omega X_1..X_n) \pm i_Y A(\omega X_1..X_n) - A((i_Y\omega)X_1..X_n) \in \mathcal{O},$$

by induction.

$\square$

E.1.1. *The Schwinger-Dyson equation for forms.*

Remark E.1.15. Now that we have a reasonable grasp of the integrands, we will proceed to define formal integrals.

Definition E.1.16. *Let (A, L) be an infinitesimal calculus, then by an integral we mean a map $I : \mathcal{I}(A, L) \to \mathbb{R}$. Such an integral is said to be of codimension k if it is zero on the homogeneous subspaces $\mathcal{I}^{l \neq -k}$. Further, given a divergence ∇, we define $\partial I := I \circ d$. In that case we define the Schwinger-Dyson equation (without boundary) for I to be $\partial I = 0$. We will see in a moment that if I is of codimension zero, then this is the usual Schwinger-Dyson equation.*

Corollary E.1.17. *Let (A, L, ∇) come from (M, μ). Let $N \subset M$ be a submanifold. Define $I_N : \mathcal{I} \to K$ by*

$$I_N(\phi) := \int_N G(\phi).$$

Then:

1. *$codim(I_N) = codim(N)$.*
2. *$\partial I_N = I_{\partial N}$.*

Remark E.1.18. This ends our discussion of the integration of forms. Our final aim in this section is to show that the formulation of the Schwinger-Dyson equation for some functional I in the geometric language reduces to the usual one in codegree zero, and to make a statement concerning the equation Schwinger-Dyson equation with fixed nonzero boundary term $\tilde{J}$ (Compare remark 5.3.3):

Theorem E.1.19. *Let I be of codimension zero. Then the following are equivalent:*

1. *$\partial I = \tilde{J}$.*
2. *$I(X(f) + \nabla(X)) = \tilde{J}(f \otimes X)$.*

Proof

First, for any I we have the identity

$$\partial I(f \otimes X) = I(di_X(f \otimes 1)) = I(di_X + i_X d)(f \otimes 1) = IL_X(f \otimes 1) = I(X(f) + \nabla(X)),$$

which proves the implication (1) $\Rightarrow$ (2). On the other hand, starting from (2), it proves that $\partial I(f \otimes X) = \tilde{J}(f \otimes X)$, so it remains to prove that $\partial I(\omega \otimes X_0..X_n) = \tilde{J}(\omega \otimes X_0..X_n)$, for $|\omega| = n$,which we prove by induction. We just saw that this is true for $n = 0$, and for $n + 1$, we note that if $|\omega| = n + 1$, then $\omega Y X_0..X_n = (-1)^{|\omega|}(i_Y \omega)X_0..X_n$, since $i_Y(\omega X_0..X_n)$ is an overflow form.

□

Remark E.1.20. For $\tilde{J} = 0$ this is the usual Schwinger-Dyson equation: Indeed, taking $\mu := e^{-S}\mu_{Leb}$, we have:$\nabla(\partial_i) = -\partial_i S$, so that $0 = I((\partial_i f - f\partial_i(S)) \otimes 1)$. The equation for more general vectorfields follows from $\nabla(fX) = X(f) + f\nabla(X)$.

E.2. **The Schwinger-Dyson equation on quotient manifolds.**

E.2.1. *Integration over quotient manifolds.*

Remark E.2.1. We will first note some properties of integration over quotient manifolds, and after that make the algebraic abstraction for the Schwinger-Dyson equation. The content of this section is a compact reformulation of formulas from [13], [17] and [18].

Definition E.2.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. By a G-structure for a volume manifold (M,μ) we will understand an action of G on M, and an element $Q_\mu \in Z^1(\mathfrak{g})$ such that:*

1. *$\pi : M \to M/G$ is a principal G-bundle.*
2. *$\forall_{A\in\mathfrak{g}}\ L_A\mu = Q_\mu(A)\mu$.*

We set $Q_\mathfrak{g}(X) := Tr(Ad(X))$. We define the background charge as $Q_{back} := Q_\mathfrak{g} + Q_\mu$.

Theorem E.2.3. *("Fadeev-Popov",[13]). Let G be a connected Lie group, and (M,μ) a G-volume manifold with $\partial M = \emptyset$. Let $\phi : M \to \mathbb{R}^{|G|}$, such that the zero set $Z(\phi)$ is an n-sheeted cover of M/G. Let $F \in L^2(\mathbb{R}^{|G|})$ such that $\int F = 1$. Let ω be a top-form on M/G, let $f : M \to \mathbb{R}$, let $\{T_i\}$ be a basis for $\mathfrak{g}$, and let $v := T_1..T_{|G|} \in \bigwedge^{max}(\mathfrak{g})$. Then:*

1. *$\int_{M/G}\omega = \frac{\pm 1}{n}\int_M \pi^*\omega F(\phi) d\phi^1 \wedge ... \wedge d\phi^{|G|}$.*
2. *$f i_v\mu$ is a basic form [14] iff $\forall_{A\in\mathfrak{g}} L_A(f) = -Q_{back}(A)f$. In that case:*
3. *$\int_{M/G}\pi_*(f i_v\mu) = \frac{\pm 1}{n}\int_M fF(\phi) det_{ij}(T_i(\phi^j))\mu$.*
4. *Let e^{-H} be the Fourier transform of F. Then there is a constant K such that*
$$\int_{M/G}\pi_*(f i_v\mu) = K\int db \int_{Ber} dc d\bar{c} \int_M f e^{-H(b)+ib_j\phi^j + i\bar{c}_i T_j(\phi^i)c^j}\mu.$$

Proof

1. Since Z is an n-sheeted cover, and since $d\pi^*\omega = \pi^* d\omega = 0$, we have: [15]
$$\int_{M/G}\omega = \frac{1}{n}\int_{Z(\phi)}\pi^*\omega = \frac{1}{n}\int_M \pi^*\omega \wedge \phi^*(Thom(M\times\mathbb{R}^{|G|}))$$
$$= \frac{\pm 1}{n}\int_M \pi^*\omega F(\phi) d\phi^1..d\phi^{|G|}.$$

[14]A form α on M is called basic iff it can be written as $\pi^*\omega$ where ω is a form on the base M/G. If G is connected this condition is equivalent to $\forall_{A\in\mathfrak{g}}\ i_A\alpha = L_A\alpha = 0$: Indeed this implies that for any vertical vectorfield $X = f^i T_i$ on M we have $L_X\alpha = di_X\alpha + i_X d\alpha = df^i \wedge i_{T_i}\alpha + f^i L_{T_i}\alpha = 0$, so that we can take $\omega := \sigma^*\alpha$, because it is then independent of the section σ. To check that $\alpha = \pi^*\omega = \pi^*\sigma^*\alpha$, we note that both sides are equal on the horizontal space spanned by σ, and since $i_A\alpha = 0$ this is enough. For more details, see H. Cartan.

[15]The Thom class of a vectorbundle $E \to M$ is a form Θ on E such that for $d\alpha = 0$, $\int_{Z(\phi)}\alpha = \int_M \alpha\wedge\phi^*\Theta$, where ϕ is a section of E. If $E = M\times V$ then the Thom class is given as a topform $Fdx^1..dx^n$ on V, normalized to $\int_V Fdx = 1$. The idea is that if we take F to be the delta function then this will reproduce exactly the zero locus of ϕ, but since $d\alpha = 0$, we can replace Θ by any other fast decreasing representative, for example a Gaussian on V.

2. Since v is of highest degree, we have $i_A i_v \mu = 0$. Further, [16]
$$L_A(f i_v \mu) = (L_A f) i_v \mu + f i_{[A,v]} \mu + f i_v L_A \mu$$
$$= (L_A f + Q_{\mathfrak{g}}(A) f + Q_\mu(A) f) i_v \mu = (L_A f + Q_{back}(A) f) i_v \mu.$$
Therefore, since $i_v \mu$ is nowhere zero, $L_A(f i_v \mu) = 0 \Leftrightarrow L_A(f) = -Q_{back}(A) f$.
3. We now specialize point (1) to the case $\omega = \pi_*(f i_v \mu)$, i.e. $\pi^* \omega = f i_v \mu$, so that the left hand side equals:
$$\frac{\pm 1}{n} \int_M f i_v \mu F(\phi) d\phi^1 .. d\phi^{|G|} = \frac{\pm 1}{n} \int_M f F(\phi) i_v (d\phi^1 .. d\phi^{|G|}) \mu = RHS.$$
4. Here we used the Fourier representation of F, and the Berezin representation of det. The variables $c, \bar{c}$ are known as Fadeev-Popov ghosts, and the function $G(\phi)$ as a gauge fixing term.

□

Definition E.2.4. *(BRST, [17]).*

1. *Define the following derivation on functions $\mathcal{O}$ of $m \in M, b, c,$ and $\bar{c}$, i.e. on $\mathcal{F}(M) = Map(M, \mathbb{R})$ tensored with the polynomial algebra on b's and the exterior algebra on $c, \bar{c}$'s:*
$$Q := c^j T_j - \frac{1}{2} f_{ij}^k c^i c^j \frac{\partial}{\partial c^k} - b_i \frac{\partial}{\partial \bar{c}_i}.$$
Here T_i is a basis element of $\mathfrak{g}$ acting on $\mathcal{F}(M)$.
2. *A function $\mathcal{O}$ will be called basic iff $Q\mathcal{O} = -Q_{back}(T_i) c^i \mathcal{O}$.*

Theorem E.2.5. *(BRST).*

1. $Q^2 = 0$.
2. *Assume H is polynomial. Then there is a Berezin-odd function Ψ, i.e. having an odd total number of c and $\bar{c}$'s, (depending on F and ϕ) and a number K such that*
$$\int_{M/G} \pi_*(f i_v \mu) = K \int db \int_{Ber} dc d\bar{c} \int_M \mu f e^{Q(\Psi)}.$$
(Ψ is known as a gauge fixing fermion).

Proof

1. Since Q is a derivation, and since Q is odd, Q^2 is also a derivation, so that it suffices to prove that Q^2 is zero on generators. Indeed:
(a)
$$Q^2(f) = Q(c^j T_j(f)) = Q(c^j) T_j(f) - c^j Q(T_j(f)) = -\frac{1}{2} f_{lm}^j c^l c^m T_j(f)$$
$$-c^j c^k T_k T_j(f) = -\frac{1}{2} c^l c^m [T_l, T_m](f) - \frac{1}{2} c^j c^k [T_k, T_j](f) = 0.$$
(b)
$$Q^2(\bar{c}_i) = Q(-b_i) = 0 = Q^2(b_i).$$

[16] If $M : V \to V$ is linear, then M acts on $\bigwedge^{max}$ by derivations. The latter action is $Tr_V(M)$ times the identity which one verifies by making M act on some fixed element $e_1 ... e_{max}$. Therefore, $[A, v] = Tr(Ad(A)).v = Q_{\mathfrak{g}}(A).v$.

(c) $Q^2(c_i) = 0$, because in that case only $-\frac{1}{2} f^k_{ij} c^i c^j \frac{\partial}{\partial c^k}$ matters, which is proportional to the Lie algebra cohomology operator.

2. Since we allow for an arbitrary constant K, we may assume that $H(0) = 0$, so that H is a polynomial of degree at least 1. But all such polynomials are Q-exact since $b_i = Q(-\bar{c}_i)$. Further,
$$Q(-i\bar{c}_j\phi^j) = -iQ(\bar{c}_j)\phi^j + i\bar{c}_j Q(\phi^j) = ib_j\phi^j + i\bar{c}_j c^k T_k(\phi^j),$$
so that we see that there is a Ψ such that $-H(b) + ib_j\phi^j + i\bar{c}_i T_j(\phi^i)c^j = Q(\Psi)$.

□

E.2.2. *Algebraic abstraction.*

Remark E.2.6. Just like we replaced the notion of a volume form by the notion of a divergence in order to find the infinite dimensional version of integration over volume manifolds, in the same way will we now make algebraic abstractions hoping to define infinite dimensional integration over quotients. What we have to avoid now is the use of any element $v \in \bigwedge^{max}(L)$, because there need not be any maximal degree. The Fadeev-Popov representation, point 4 in theorem E.2.3, is more practicable in the infinite dimensional case, because it only involves summation over an infinite basis, for example in $T_j(\phi^i)c^j$. It is outside the scope of this work to consider regularizations of this summation, so from now on we assume that this kind of summation is possible. Thus, for example we also assume that $X \mapsto Tr(Ad(X))$ exists.

Definition E.2.7. *A $\mathfrak{g}$-symmetric formal volume manifold is defined to be a combination $(A, L, \nabla, Q_\mu, \mathfrak{g})$, where*

1. *(A, L, ∇) is a formal volume manifold.*
2. *$\mathfrak{g} \leq L$ is a sub Lie algebra, called the zero-mode or gauge algebra.*
3. *$Q_\mu \in Z^1(\mathfrak{g})$.*
4. *$\forall_{n\in\mathfrak{g}}\ \nabla(n) = Q_\mu(n)$.*

Further, we set $Q_\mathfrak{g}(X) := Tr(Ad(X))$, and $Q_{back} := Q_\mathfrak{g} + Q_\mu$. For Ψ a Berezin-odd element, $D \in \{T_j, \frac{\partial}{\partial c^k}, \frac{\partial}{\partial \bar{c}_i}, \frac{\partial}{\partial b_i}\}$, and with $\nabla(\frac{\partial}{\partial c^k}) := \nabla(\frac{\partial}{\partial \bar{c}_i}) := \nabla(\frac{\partial}{\partial b_i}) := 0$, the Schwinger-Dyson equation is defined as:
$$\langle D(\mathcal{O}) + \nabla(D)\mathcal{O} + (-1)^{\mathcal{O}D}\mathcal{O}DQ(\Psi)\rangle = 0,$$
for maps $\mathcal{O} \mapsto \langle\mathcal{O}\rangle$, where $\mathcal{O}$ is a combination of functions on M and functions of $c, \bar{c}, b$.

Remark E.2.8. As in the usual case, the above Schwinger-Dyson equation is motivated by the same property of
$$\mathcal{O} \mapsto \langle\mathcal{O}\rangle := \int dbdcd\bar{c}\mu\mathcal{O}e^{Q(\Psi)}.$$
As we have seen using Thom's theorem, the Fadeev-Popov integral does not depend on the choice of ϕ or F, since the quotient integral does not involve these objects. An alternative proof is given in the following theorem:

Theorem E.2.9. *(Gauge invariance using the Schwinger-Dyson equation.)*

1. $\langle Q\mathcal{O}\rangle = -\langle Q_{back}(T_i)c^i\mathcal{O}\rangle$.
2. *If $\mathcal{O}_1$ is basic, then $\langle\mathcal{O}_1 Q(\mathcal{O}_2)\rangle = 0$.*

3. *Let $t \mapsto \Psi_t$ be a family of Berezin-odd elements. Let $\langle . \rangle_\Psi$ be a family of solutions of the corresponding Schwinger-Dyson equations satisfying $\partial_t \langle \mathcal{O} \rangle_{\Psi_t} = \langle \mathcal{O} Q \partial_t \Psi_t \rangle_{\Psi_t}$, and let $\mathcal{O}$ be basic, then $\partial_t \langle \mathcal{O} \rangle_{\Psi_t} = 0$.*

Proof

1.
$$\langle Q\mathcal{O} \rangle = \langle c^j T_j \mathcal{O} - \frac{1}{2} f^k_{ij} c^i c^j \frac{\partial \mathcal{O}}{\partial c^k} - b_i \frac{\partial \mathcal{O}}{\partial \bar{c}_i} \rangle$$
$$= \langle T_j(c^j \mathcal{O}) - \frac{\partial}{\partial c^k}(\frac{1}{2} f^k_{ij} c^i c^j \mathcal{O}) + \frac{\partial}{\partial c^k}(\frac{1}{2} f^k_{ij} c^i c^j)\mathcal{O} - \frac{\partial}{\partial \bar{c}_i}(b_i \mathcal{O}) \rangle$$
$$= \langle -c^j \mathcal{O} T_j Q\Psi + \frac{1}{2} f^k_{ij} c^i c^j \mathcal{O}(-1)^{\mathcal{O}} \frac{\partial}{\partial c^k} Q\Psi + b_i \mathcal{O}(-1)^{\mathcal{O}} \frac{\partial}{\partial \bar{c}_i} Q\Psi \rangle$$
$$+ \langle -c^j \mathcal{O} \nabla(T_j) + \frac{1}{2} f^k_{kj} c^j \mathcal{O} - \frac{1}{2} f^k_{ik} c^i \mathcal{O} \rangle$$
$$= \langle -(-1)^{\mathcal{O}} \mathcal{O} \{ c^j T_j - \frac{1}{2} f^k_{ij} c^i c^j \frac{\partial}{\partial c^k} - b_i \frac{\partial}{\partial \bar{c}_i} \} Q\Psi \rangle$$
$$+ \langle -c^j \mathcal{O} Q_\mu(T_j) - Q_{\mathfrak{g}}(T_i) c^i \mathcal{O} \rangle$$
$$= \langle -(-1)^{\mathcal{O}} \mathcal{O} Q^2 \Psi \rangle - \langle (Q_\mu + Q_{\mathfrak{g}})(T_i) c^i \mathcal{O} \rangle = -\langle Q_{back}(T_i) c^i \mathcal{O} \rangle.$$

2.
$$\langle \mathcal{O}_1 Q(\mathcal{O}_2) \rangle = (-1)^{\mathcal{O}_1} (\langle Q(\mathcal{O}_1 \mathcal{O}_2) \rangle - \langle (Q\mathcal{O}_1)\mathcal{O}_2 \rangle)$$
$$= (-1)^{\mathcal{O}_1} \langle -Q_{back}(T_i) c^i \mathcal{O}_1 \mathcal{O}_2 \rangle + (-1)^{\mathcal{O}_1} \langle Q_{back}(T_i) c^i \mathcal{O}_1 \mathcal{O}_2 \rangle = 0.$$

3. By taking $\mathcal{O}_2 := \partial_t \Psi_t$ in the previous point.

□

Example E.2.10. (Maxwell-Feynman). Consider the set $M = \Omega^1(\mathbb{R}^D)$ of connections on a product bundle $\mathbb{R}^D \times U(1)$ over $\mathbb{R}^D$. Let G be $\Omega^0(\mathbb{R}^D)$, which acts on M by $\varphi : A \mapsto A + d\varphi$. The weight $S(A) := \int_{\mathbb{R}^D} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} dx$, where $F = dA$ is well-defined on the quotient M/G. Let us try to compute

$$\int_{M/G} f(A) e^{-S(A)} DA,$$

where DA means that we use the affine structure of M to produce some preferred vectorfields $\delta/\delta A(x)$ and then use the Schwinger-Dyson equation. Since we go to the quotient, we need a gauge fixing function ϕ, for which we take $\phi^x(A) := \partial_\mu A^\mu(x)$, together with a condition at infinity to ensure that no remaining group elements with $\Delta\varphi = 0$ leave the condition $\phi(A) = 0$ invariant. Then the functions $T_j(\phi^i)$ are A-independent:

$$\partial_t \phi^x(A + td\varphi) = \Delta\varphi(x),$$

So that the term $X(T_j(\phi^i))$ does not enter the Schwinger-Dyson equation. For F, let us take a Gaussian bump function:

$$F(\phi) := \exp(-\frac{1}{2} \int_{\mathbb{R}^D} \phi^x \phi^x dx).$$

What we see then, is that the quotient calculation is reduced to the calculation of $\int_M e^{-S_1(A)} f(A) DA$, where

$$S_1(A) := \int_{\mathbb{R}^D} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu A^\mu)^2.$$

This is a Gaussian weight and we can now even integrate the function $A \mapsto A^\mu(x)$, even if it is not G-invariant. Since $\frac{\delta S_1}{\delta A^\mu(x)} = -\Delta A_\mu(x)$, we see that $\langle A_\mu(x) A_\nu(y)\rangle = -g_{\mu\nu} f_D(x-y)$, where the functions f_D were defined in section B.1.2. From this we can also read off the contractions for G-invariant objects like $F_{\mu\nu}(x)$, or $I_L(A) := \oint_L A$:

$$\langle I_L I_M\rangle = -\oint_L dx^\mu \oint_M dy^\nu g_{\mu\nu} f_D(x-y).$$

Remark E.2.11. It was noted by Batalin and Vilkovisky [18] that the BRST expression

$$Q := c^j T_j - \frac{1}{2} f_{ij}^k c^i c^j \frac{\partial}{\partial c^k} - b_i \frac{\partial}{\partial \bar{c}_i}$$

used in theorem E.2.5 can be generalized by replacing it by a more general polynomial in the symbols $f(m), c, \bar{c}, b, f_i^* := T_i, c_i^* := \frac{\partial}{\partial c^i}, \bar{c}^{i*} := \frac{\partial}{\partial \bar{c}_i}, b^{i*} := \frac{\partial}{\partial b_i}$. Just like in the BRST case where Q occurs in the integral only through $Q(\Psi)$, one is led to:

1. Assign Berezin parities to the starred variables as if they are evaluated on the odd element Ψ:
$$Parity(\phi_\alpha^*) := Parity(\frac{\partial \Psi}{\partial \phi^\alpha}) = Parity(\phi^\alpha) + 1.$$
2. Consider the BRST integral, but now with the more general Q. In that case $Q(\Psi)$ means replacing ϕ_α^* by $\frac{\partial \Psi}{\partial \phi^\alpha}$ in Q.
3. Find a condition on Q such that the end-result is independent of Ψ.

The fact that the Q of BRST satisfies $Q^2 = 0$ can now be reformulated as the fact that $\{Q, Q\} = 0$, where $\{.,.\}$ is the Schouten bracket. For more general Q's there is also a condition that will guarantee independence of Ψ, known as the BV master equation [18, formula 16], but it does not read $\{Q, Q\} = 0$. Instead, there is an additional term in the equation, which is zero in the special BRST-case.

Appendix F. Reference material on some natural super Lie algebras.

Definition F.0.12. *Let $(G, +)$ be an abelian group, and let $< .,. >: G \times G \to \mathbb{Z}_2$ be bilinear symmetric.*

1. *A G-graded vectorspace or G-vectorspace is a vectorspace V together with a direct sum decomposition $V = \oplus_{g \in G} V_g$. For homogeneous elements $a, b, .. \in V$, we will denote the degree by $|a|, |b|, .. \in G$. The number $(-1)^{<|a|,|b|>}$ will be denoted simply by $(-1)^{ab}$. Thus, for example, we have $(-1)^{a(b+c)} = (-1)^{ab+ac}$, whereas $(-1)^{abc}$ is undefined.*
2. *A G-algebra of degree $w \in G$ is an algebra of which the underlying vectorspace is G-graded, and such that the composition is from $A_g \otimes A_h$ to A_{g+h+w}. By setting $A'_g := A_{g-w}$, we get a G-algebra of degree 0. Thus, if one considers only one algebraic structure at a time, one may restrict to degree zero, which we will do from now on.*
3. *A linear map $M : A \to B$ is said to be of degree $|M| \in G$ iff it maps A_h to $B_{h+|M|}$. We write $(-1)^{M.}$ for $(-1)^{<|M|,.>}$.*
4. *The above notion of G-gradation does not depend on $< .,. >$. We will now introduce some concepts that do depend on $< .,. >$, and this situation is usually referred to as "super". We will stop mentioning the dependence on G: Thus, algebra means G-graded algebra. In most cases, we have $G = \mathbb{Z}$ and $< g, h >:= gh \bmod 2\mathbb{Z}$.*
5. *M as above is called a derivation w.r.t. $< .,. >$ or a superderivation iff $M(ab) = M(a)b + (-1)^{Ma} aM(b)$.*
6. *An algebra is called symmetric (w.r.t. to $< .,. >$) or supersymmetric, iff $ab = (-1)^{ab} ba$.*
7. *It is antisymmetric iff $ab = -(-1)^{ab} ba$.*
8. *An algebra is called associative iff $a(bc) = (ab)c$.*
9. *We will now also drop the word "super": It will be understood whenever a definition depends on $< .,. >$.*
10. *An algebra is called pre-Lie* [10] *iff $a(bc) - (-1)^{ba} b(ac) = (ab)c - (-1)^{ab}(ba)c$. Pre-Lie compositions will usually be denoted as $[a \rhd b]$, in view of their connection with subgaussian contraction algebras.*
11. *An algebra is called Jacobi iff $\sum_{Cycl(a,b,c)} (-1)^{ac} a(bc) = 0$.*
12. *An algebra is called deriving iff $a(bc) = a(bc) + (-1)^{ab} b(ac)$.*
13. *An algebra is called Lie iff it is antisymmetric and Jacobi, or equivalently antisymmetric and deriving. A Lie composition will be denoted by $[.,.]$ or $\{.,.\}$.*
14. *An algebra with two compositions is called Poisson iff the first is symmetric associative with unit, the second, denoted by $\{.,.\}$ is Lie, and the two are compatible in the sense that $\{a, bc\} = \{a, b\}c + (-1)^{bc}\{a, c\}b$*
15. *A Batalin-Vilkovisky algebra A is a Poisson algebra of which the associative product is of degree 0 and the bracket of degree -1, together with a differential $\partial : A \to A$, i.e. $\partial^2 = 0$, of degree -1, such that:*
$$\{a, b\} = a(\partial b) + (-1)^{|a|}(\partial a)b - (-1)^{|a|}\partial(ab).$$
See Getzler, [26].

Remark F.0.13. Proofs of identities that hold in the usual case where $< .,. >= 0$ need in general not be repeated in the supercase. Indeed, if the ungraded identity

concerns polynomial expressions in a number of variables $a_1, ..., a_n \in A$, say

$$[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [a_2, [a_1, a_3]],$$

then the above definitions are such that one gets the corresponding super identity by multiplying each seperate term with ± 1, according to the permutation that the variables have undergone on paper w.r.t., say, the order $a_1 a_2 a_3$; so in the above case this gives the expression

$$[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [a_2, [a_1, a_3]](-1)^{a_1 a_2}.$$

One may thus consider the first formula to be shorthand notation for the second, and in this way, one can consider the proofs of ungraded identities to be shorthand notation for the supercase, since in every expression in the proof one may add the corresponding signs. For example this reasoning applies to the proof that the semi-direct product defined below produces a new super Lie algebra as claimed: First prove this fact in the ungraded case, and then graded proof is produced by adding minus signs according to the permutation rule.

We list a number of functors between these categories.

1. *Direct sums and tensorproducts of G-spaces are G-graded.*
2. *The tensoralgebra of V is defined as $T(V) := \oplus_n V^{\otimes n}$.*
3. *The symmetric algebra is defined as $Sym(V) := T(V)/I$, where I is the two-sided ideal generated by elements of the form $ab - (-1)^{ab} ba$. It is an associative $\mathbb{Z} \oplus G$ algebra with unit containing V, such that $v \in V$ has degree $(1, |v|)$ in $Sym(V)$. It is symmetric w.r.t. the form $(n \oplus g, m \oplus h) :=< g, h >$.*
4. *The exterior algebra is defined as $\bigwedge(V) := T(V)/J$, where J is the two-sided ideal generated by elements of the form $ab + (-1)^{ab} ba$. It is a $\mathbb{Z} \oplus G$-graded algebra, symmetric w.r.t. the form $(n \oplus g, m \oplus h) := nm + < g, h >$.*
5. *Direct sums of G-algebras are again G-algebras.*
6. *The tensorproduct of two associative algebras is understood to have the following associative multiplication (which depends on $< .,. >$): $(a \otimes b)(c \otimes d) := (-1)^{bc}(ac \otimes bd)$.*
7. *The tensor product of two Poisson algebras is again Poisson using the above associative multiplication and bracket*
   $$\{a_1 \otimes b_1, a_2 \otimes b_2\} := (-1)^{a_2 b_1}\{a_1, a_2\} \otimes b_1 b_2 + (-1)^{a_2 b_1} a_1 a_2 \otimes \{b_1, b_2\}.$$
8. *Further, we define Associative$\otimes$Lie to be Lie as follows: $[a \otimes X, b \otimes Y] := (-1)^{bX} ab \otimes [X, Y]$.*
9. *The semi-direct product of two Lie algebras A,B w.r.t. $\pi : A \to Der(B)$ (degree preserving) is defined as the vectorspace $A \oplus B$, with composition $[a_1 \oplus b_1, a_2 \oplus b_2] := [a_1, a_2] \oplus [b_1, b_2] + \pi(a_1)(b_2) - (-1)^{b_1 a_2} \pi(a_2)(b_1)$.*
10. *Associative$\to$Pre Lie; $[a \triangleright b] := ab$.*
11. *Pre-Lie $\to$ Lie; $[a, b] := [a \triangleright b] - (-1)^{ab}[b \triangleright a]$. In particular, the inner endomorphisms $I_a(b) := [a \triangleright b]$ form a representation of the Lie algebra, since by definition of pre Lie, we have $[I_a, I_b] = I_{[a,b]}$.*
12. *Associative $\to$ Lie; $A \mapsto Der(A)$. (The derivations are closed under the commutator bracket.)*
13. *Lie$\to$Poisson; $L \mapsto Sym(L)$, with obvious associative product, and symmetric Schouten bracket $\{X_1..X_n, Y_1..Y_m\} := \sum_{i,j} X_{[1,n]\backslash i}[X_i, Y_j] Y_{[1,m]\backslash j}$. (With signs added in supercase.)*

14. *Lie→BV;* $L \mapsto \bigwedge(L)$*, with the antisymmetric Schouten bracket:*
$$[X_1..X_n, Y_1..Y_m] := \sum_{i,j}(-1)^{i+j}[X_i, Y_j]X_{[1,n]\setminus i}Y_{[1,m]\setminus j},$$
and differential
$$\partial(X_1..X_n) := \sum_{i<j}(-1)^{i+j+1}[X_i, X_j]X_{[1,n]\setminus ij}.$$
15. *Whenever* $d : G \to \mathbb{Z}$ *is a group homomorphism, we can extend a* G*-Lie algebra with a "counting" element* N_d *of* G*-degree* 0*, by setting* $[N_d, N_d] := 0$ *and* $[N_d, a] := -[a, N_d] := d(|a|)a$*.*

Definition F.0.14. *Let* L *be a Lie algebra. To it we associate* $Car(L)$*, the Cartan algebra of* L*, which is is a* $\mathbb{Z}$*-Lie algebra generated by symbols* d, i_X, L_X*, linear in* $X \in L$*, with degrees and relations as in the following table* [17] *:*

$[.,.]$	(deg)	d	i_Y	L_Y
d	(1)	0	L_Y	0
i_X	(-1)		0	$i_{[X,Y]}$
L_X	(0)			$L_{[X,Y]}$

We will see in a minute that this is a Lie algebra. Further, there is a natural map $K : Car(L) \to End(\bigwedge(L))$*, as follows:*

1. $d(X_1..X_n) := \sum_{i<j}(-1)^{i+j+1}[X_i, X_j]X_{[1,n]\setminus ij}$,
2. $i_X(X_1..X_n) := XX_1..X_n$.
3. $L_X(X_1..X_n) := \sum_{i=1}^{n} X_1..[X, X_i]..X_n$.

Theorem F.0.15. $[K(A), K(B)] = K([A, B])$.

Proof

We will prove this in the order ii, id, Li, LL, dd, Ld:

1. $[i_X, i_Y](Z_1..Z_n) = (XY + YX)Z_1..Z_n = 0$.
2. $[i_X, d](Z_1..Z_n) = Xd(Z_1..Z_n) + d(XZ_1..Z_n)$
$$= X\sum_{i<j}(-1)^{i+j+1}[Z_i, Z_j]Z_{[1,n]\setminus ij} + \sum_{j=1}^{n}(-1)^{j+1}[X, Z_j]Z_{[1,n]\setminus j}$$
$$+\sum_{i<j}(-1)^{i+j+1}[Z_i, Z_j]XZ_{[1,n]\setminus ij} = L_X(Z_1..Z_n).$$
3. $[L_X, i_Y](Z_1..Z_n) = [X, YZ_1..Z_n] - Y[X, Z_1..Z_n]$
$$= [X, Y]Z_1..Z_n = i_{[X,Y]}Z_1..Z_n.$$
4. $X \mapsto L_X$ is the adjoint representation on $\bigwedge(L)$.
5. We prove this together with
6. First define $\Delta := [d, d]$, and $R_X := [L_X, d]$.

Lemma F.0.15.1. $[i_X, R_Y] = 0$*, and* $[i_X, \Delta] = 2R_X$*.*

Proof

Using the known commutators involving i_X, we have:

[17] More generally, one can make a $\mathbb{Z}\oplus G$-Lie algebra from a G algebra, by giving i_X, say, degree $(-1, |X|)$. This is what happens if you want to have a Cartan calculus on a supermanifold.

(a) $[i_X, R_Y] = [i_X, [L_Y, d]] = [[i_X, L_Y], d] + [L_Y, [i_X, d]] = [i_{[X,Y]}, d] + [L_Y, L_X] = L_{[X,Y]} + L_{[Y,X]} = 0$, and
(b) $[i_X, \Delta] = [i_X, [d, d]] = [[i_X, d], d] - [d, [i_X, d]] = [L_X, d] - [d, L_X] = 2R_X$.

□

This allows us to prove that $\Delta(Z_1..Z_n) = R_X(Z_1..Z_n) = 0$ by induction on n: The case $n = 0$ is clear since $d(1) = L_X(1) = 0$. Assume the statement to be true up to n, then:
(a) $\Delta(YZ_1..Z_n) = [\Delta, i_Y](Z_1..Z_n) = 2R_Y(Z_1..Z_n) = 0$, and
(b) $R_X(YZ_1..Z_n) = [R_X, i_Y](Z_1..Z_n) = 0$.

□

Theorem F.0.16. *$Car(L)$ is a super Lie algebra.*

Proof
One may check this directly, but we will just prove it by intimidation: In order to prove that an algebra is a Lie algebra, it suffices to find a faithful representation. We already have a representation, but it need not be faithful, since for example if L is Abelian, then d acts as 0. To that end, let V be the vectorspace underlying the Lie algebra L, and let $\tilde{L}$ be the free Lie algebra over V. Then the map $\tilde{\rho} : Car(\tilde{L}) \to End(\bigwedge(\tilde{L}))$ is injective: One can see this by noting that all the operators have different degree, and the operator L_{X-Y} is zero iff $X - Y = 0$. So $\tilde{\rho}$ is injective, and therefore $Car(\tilde{L})$ is a super Lie algebra. But there is a surjective morphism $Car(\tilde{L}) \to Car(L)$, so $Car(L)$ is super Lie.

□

Theorem F.0.17. *Using the above module, we can make some other modules:*

1. *$Car(L)$ is represented on $\bigwedge(L)^{dual}$ by $d\omega := -\omega \circ d$, $i_X\omega := \omega \circ i_X$, and $L_X\omega := -\omega \circ L_X$.*
2. *The subspace $\bigwedge(L)^{dual,[0,n]} \oplus B^{n+1}(L) \subset \bigwedge(L)^{dual}$ is a submodule, where B^{n+1} denotes the exact elements α of degree $n+1$, i.e. such that $\exists_\beta\ \alpha = d\beta$.*

Proof

1. Denote the operators on $\bigwedge(L)$ by d, i_X, L_X, and those on the dual by $*d, *i_X, *L_X$. Then $*d = -d^T$, $*i_X = i_X^T$, and $*L_X = -L_X^T$, where $A^T\omega := \omega \circ A$. using $[A^T, B^T] = [B, A]^T$, both for commutators and anti-commutators, we have:
   (a) $[*i_X, *i_Y] = [i_X^T, i_Y^T] = [i_Y, i_X]^T = 0$.
   (b) $[*i_X, *d] = [-d, i_X]^T = -L_X^T = *L_X$.
   (c) $[*L_X, *i_Y] = [i_Y, -L_X]^T = -i_{[Y,X]}^T = *i_{[X,Y]}$.
   (d) $[*L_X, *L_Y] = [-L_Y, -L_X]^T = L_{[Y,X]}^T = *L_{[X,Y]}$.
   (e) $[*d, *d] = [-d, -d]^T = 0$.
   (f) $[*d, *L_X] = [-L_X, -d]^T = 0$.
2. It is closed under i_X because i_X lowers degree and annihilates zero degree. It is closed under L_X, because L_X preserves degree and commutes with d. It is closed under d, because it increases degree by 1, maps $\bigwedge^n(L)$ to B^{n+1}, and maps B^{n+1} to zero, since $d^2 = 0$.

□

Definition F.0.18. *Associated to any module V of $Car(L)$ is associated a new Lie algebra $Car(L,V)$, namely the semi-direct product of $Car(L)$ and V, where V is seen as an Abelian algebra. Thus, the extension looks as follows, with $v \in V$:*

$[.,.]$	(deg)	d	i_Y	L_Y	M_v
d	(1)	0	L_Y	0	M_{dv}
i_X	(-1)		0	$i_{[X,Y]}$	$M_{i_X v}$
L_X	(0)			$L_{[X,Y]}$	$M_{L_X v}$
M_w	$(\|w\|)$				0

We have already constructed a number of natural modules, so we get a number of larger super Lie algebras associated to L, for example $Car(L, \bigwedge(L)^{dual})$, or

$$Car^{(n)}(L) := Car(L, \bigwedge^{\leq n}(L)^{dual} \oplus B^{n+1}).$$

Of special interest is an algebra which we choose to call the $BRST$-algebra of L, which is $Car^{(1)}(L)$, extended with a counting element N:

Theorem F.0.19. *Explicitly, $BRST(L)$ is isomorphic to an algebra generated by symbols $N, d, i_X, L_X, c_\alpha, e_\alpha, 1$, where $X \in L$ and $\alpha \in L^{dual}$, with the following composition:*

$[.,.]$	(deg)	N	d	i_Y	L_Y	c_β	e_β	1
N	(0)	0	d	$-i_Y$	0	c_β	$2e_\beta$	0
d	$(+1)$		0	L_Y	0	e_β	0	0
i_X	(-1)			0	$i_{[X,Y]}$	$\beta(X)1$	$c_{[X,\beta]}$	0
L_X	(0)				$L_{[X,Y]}$	$c_{[X,\beta]}$	$e_{[X,\beta]}$	0
c_α	$(+1)$					0	0	0
e_α	$(+2)$						0	0
1	(0)							0

Here $[X,\beta]$ denotes the coadjoint action: $[X,\beta](Y) := -\beta([X,Y])$.

Proof
Define $1 := M_1$, and for $\alpha \in L^{dual}$, set $c_\alpha := M_\alpha$, and $e_\alpha := M_{d\alpha}$. Then, since we already know the commutation relations of $Car(L)$ itself, and the that of N, it remains to check that:

1. $[d, c_\beta] = [d, M_\beta] = M_{d\beta} = e_\beta$.
2. $[d, e_\beta] = [d, M_{d\beta}] = 0$.
3. $[i_X, c_\beta] = [i_X, M_\beta] = M_{i_X\beta} = \beta(X)1$.
4. $[i_X, e_\beta] = [i_X, M_{d\beta}] = M_{i_X d\beta} = c_{i_X d\beta}$, and $i_X d\beta(Y) = d\beta(X,Y) = -\beta([X,Y]) = [X,\beta](Y)$.
5. $[L_X, c_\beta] = [L_X, M_\beta] = M_{L_X\beta} = c_{L_X\beta}$, and $L_X\beta(Y) = -\beta([X,Y]) = [X,\beta](Y)$.
6. All commutators among $1, c$, and e are of the form $[M_1, M_2]$ and therefore zero.

□

Corollary F.0.20. *A number of subalgebras can be seen:*

1. *First of course $Car(L) \subset BRST(L)$.*
2. *Next $bc(L) \subset BRST(L)$, where $bc(L)$ is the subalgebra generated by i_X, c_α, and 1.*

3. $Weil(L) \subset UEA(BRST(L))$, *where* $Weil(L) := \bigwedge(L^{dual}) \otimes Sym(L^{dual})$, *as follows:* $\alpha_1..\alpha_n \otimes \beta_1..\beta_m \mapsto c_{\alpha_1}..c_{\alpha_n} e_{\beta_1}..e_{\beta_m}$.

Theorem F.0.21. *We can extend the action* $L \to End(\bigwedge(L))$ *to* $BRST(L) \to End(\bigwedge(L))$ *as follows:*

1. $N(X_1..X_n) := -n.X_1..X_n$.
2. $c_\alpha(X_1..X_n) := \sum_{i=1}^n (-1)^{i+1}\alpha(X_i)X_{[1,n]\backslash i}$.
3. $e_\alpha(X_1..X_n) := \sum_{i<j}(-1)^{i+j+1}\alpha([X_i, X_j])X_{[1,n]\backslash ij}$.
4. $1(X_1..X_n) := X_1..X_n$.

Proof

(1) We will prove the commutation relations in the order of the superscripts indicated below:

$[.,.]$	N	d	i_Y	L_Y	c_β	e_β
N	0^{16}	d^{17}	$-i_Y^{18}$	0^{19}	c_β^{20}	$2e_\beta^{21}$
d		0^6	L_Y^2	0^5	e_β^{11}	0^{12}
i_X			0^1	$i_{[X,Y]}^3$	$\beta(X)1^7$	$c_{[X,\beta]}^9$
L_X				$L_{[X,Y]}^4$	$c_{[X,\beta]}^8$	$e_{[X,\beta]}^{15}$
c_α					0^{10}	0^{13}
e_α						0^{14}

1. -(6) were already proved in theorem F.0.15.
7. $[i_X, c_\beta](Z_1..Z_n) = Xc_\beta(Z_1..Z_n) + c_\beta(XZ_1..Z_n) = \sum_{i=1}^n(-1)^{i+1}\beta(Z_i)XZ_{[1,n]\backslash i}$
$$+\beta(X)Z_1..Z_n + \sum_{i=1}^n(-1)^i\beta(Z_i)XZ_{[1,n]\backslash i} = \beta(X).1(Z_1..Z_n).$$
8. Both L_X and c_β are superderivations. Therefore we only need to prove the identity on (Z). Indeed:
$$[L_X, c_\beta](Z) = L_Xc_\beta(Z) - c_\beta L_X(Z) = L_X(\beta(Z)) - c_\beta([X,Z])$$
$$= -\beta([X,Z]) = [X,\beta](Z) = c_{[X,\beta]}(Z).$$
9. $[i_X, e_\beta](Z_1..Z_n) = Xe_\beta(Z_1..Z_n) - e_\beta(XZ_1..Z_n)$
$$= X\sum_{i<j}(-1)^{i+j+1}\beta([Z_i,Z_j])Z_{[1,n]\backslash ij} - \sum_{j=1}^n(-1)^{j+1}\beta([X,Z_j])Z_{[1,n]\backslash j}$$
$$-\sum_{i<j}(-1)^{i+j+1}\beta([Z_i,Z_j])XZ_{[1,n]\backslash ij} = \sum_{i=1}^n(-1)^{j+1}[X,\beta](Z_j)Z_{[1,n]\backslash j}$$
$$= c_{[X,\beta]}(Z_1..Z_n).$$
10. All c_α's are superderivations. Thus it suffices to check that
$$[c_\alpha, c_\beta](Z) = c_\alpha c_\beta(Z) + c_\beta c_\alpha(Z) = \beta(Z)c_\alpha(1) + \alpha(Z)c_\beta(1) = 0.$$

11. Let $r_\alpha := [d, c_\alpha]$. Then $[i_X, r_\alpha - e_\alpha] = 0$, since

$$[i_X, r_\alpha] = [i_X, [d, c_\alpha]] = [L_X, c_\alpha] - [d, [i_X, c_\alpha]]$$

$$= c_{[X,\alpha]} - [d, 1]\alpha(X) = c_{[X,\alpha]} = [i_X, e_\alpha].$$

Thus to prove that $r_\alpha = e_\alpha$, using the same reasoning as in lemma F.0.15.1, it remains to be shown that $r_\alpha(1) = e_\alpha(1)$. Indeed, $e_\alpha(1) = 0$ by definition, and $r_\alpha(1) = 0$ because $d(1) = 0$ and $c_\alpha(1) = 0$.
12. $[d, e_\alpha] = [d, [d, c_\alpha]] = [[d, d], c_\alpha] - [d, [d, c_\alpha]] = -[d, e_\alpha] \quad \Rightarrow [d, e_\alpha] = 0$.
13. This we will prove simultaneously with
14. Set $r_{\alpha\otimes\beta} := [c_\alpha, e_\beta]$; $q_{\alpha\otimes\beta} := [e_\alpha, e_\beta]$.

Lemma F.0.21.1. $[i_X, r_{\alpha\otimes\beta}] = 0$; $[i_X, q_{\alpha\otimes\beta}] = r_{[X,\alpha]\otimes\beta - [X,\beta]\otimes\alpha}$.

Proof

(a) $[i_X, r_{\alpha\otimes\beta}] = [i_X, [c_\alpha, e_\beta]] = [[i_X, c_\alpha], e_\beta] - [c_\alpha, [i_X, e_\beta]]$

$$= [\alpha(X)1, e_\beta] - [c_\alpha, c_{[X,\beta]}] = 0.$$

(b) $[i_X, q_{\alpha\otimes\beta}] = [i_X, [e_\alpha, e_\beta]] = [[i_X, e_\alpha], e_\beta] + [e_\alpha, [i_X, e_\beta]]$

$$= [c_{[X,\alpha]}, e_\beta] + [e_\alpha, c_{[X,\beta]}] = r_{[X,\alpha]\otimes\beta} - r_{[X,\beta]\otimes\alpha}.$$

□

Again, like in lemma F.0.15.1, it suffices to prove that $r_{\alpha\otimes\beta}(1) = q_{\alpha\otimes\beta}(1) = 0$. This is true because $c_\alpha(1) = e_\alpha(1) = 0$.
15. $[L_X, e_\beta] = [L_X, [d, c_\beta]] = [d, [L_X, c_\beta]] = [d, c_{[X,\beta]}] = e_{[X,\beta]}$.
16. to (21): These commutators follow from the fact that if $A : \bigwedge^k \to \bigwedge^{k+n}$, then $[N, A] = -nA$: $[N, A](Z_1..Z_k) = NA(Z_1..Z_k) - AN(Z_1..Z_k)$

$$= -(n+k)A(Z_1..Z_k) + A(kZ_1..Z_k) = -nA(Z_1..Z_k).$$

□

Remark F.0.22. Finally we include some often used theorems in the Hamiltonian approach to BRST symmetry. Note however still the similarity between $H(d)$ below, and Q in definition E.2.4.

Theorem F.0.23. *(Associative BRST construction). If L is finite dimensional with basis $\{T_a\}$, and dual basis $\{T^a\}$, then there is a Lie algebra morphism $H : BRST(L) \to UEA(L) \otimes UEA(bc(L))$, where the right-hand side is regarded as an associative algebra, as follows:*

1. $H(N) := -1 \otimes i_{T_a} c_{T^a}$.
2. $H(d) := T_a \otimes c_{T^a} - 1 \otimes \frac{1}{2} i_{[T_a,T_b]} c_{T^a} c_{T^b}$.
3. $H(i_X) := 1 \otimes i_X$.
4. $H(L_X) := X \otimes 1 + 1 \otimes i_{[X,T_a]} c_{T^a}$.
5. $H(c_\alpha) := 1 \otimes c_\alpha$.
6. $H(e_\alpha) := -\frac{1}{2}\alpha([T_a, T_b]) c_{T^a} c_{T^b}$.
7. $H(1) := 1 \otimes 1$.

In that case the representation on $\bigwedge(L)$ factorises through this homomorphism.

Corollary F.0.24. *Let L be a finite dimensional Lie algebra. If L is represented on V, and $bc(L)$ is represented on W, then $BRST(L)$ is naturally represented on $V \otimes W$. Indeed*

$$\begin{aligned} BRST(L) &\to UEA(L) \otimes UEA(bc(L)) \to End(V) \otimes UEA(bc(L)) \\ &\to End(V) \otimes End(W) \to End(V \otimes W). \end{aligned}$$

Theorem F.0.25. *(Poisson BRST construction.) Let L be a finite dimensional Lie algebra, Then there is a super natural Lie homomorphism $H : BRST(L) \to Sym(L) \otimes Sym(bc(L))$, as follows: ($\{T_a\}$ is a basis for L, and $\{T^a\}$ its dual basis):*

1. $H(1) := 1 \otimes 1$
2. $H(i_X) := 1 \otimes i_X$
3. $H(c_\alpha) := 1 \otimes c_\alpha$
4. $H(N) := 1 \otimes c_{T^a} i_{T_a}$
5. $H(d) := T_a \otimes c_{T^a} - \frac{1}{2} c_{T^a} c_{T^b} i_{[T_a, T_b]}$
6. $H(L_X) := X \otimes 1 + c_{T^a} i_{[T_a, X]}$
7. $H(e_\alpha) := -\frac{1}{2} c_{T^a} c_{T^b} \alpha([T_a, T_b])$

Example F.0.26. The above is often seen in the following situation:

1. Let $\mathcal{F}$ be a Poisson algebra with 1, and let $h : L \to \mathcal{F}$ be a Lie homomorphism. Then we get a Lie homomorphism
$$BRST(L) \to \mathcal{F} \otimes Sym(bc(L)).$$
2. Let $\mathcal{F}$ be the functions on a symplectic manifold M. Then the Poisson algebra $\mathcal{F} \otimes Sym(bc(L))$ is usually referred to as the functions on extended phase space. In this language the above theorem says that if phase space M is endowed with a L-hamiltonian, then extended phase space has a natural $BRST(L)$-hamiltonian.

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